Research Workshop
Calculations for Modern and Future Colliders
July 9-23, Dubna, BLTP

Lecture Course: "Standard Model and its application for precision experiments at Z-resonance"

OUTLINE

1. Introduction:
   - Prelude:
     Standard Model (SM) – an example of QFT – a tool for precision calculations in modern High Energy Physics (HEP);
   - Quantum fields for particles in the SM;
   - A QFT reminder;
   - The notion of Input Parameter Set (IPS);
   - QED Lagrangian and Feynman rules.
2. Standard Model Lagrangian building:
   - SM Lagrangian in R_ξ gauge, gauge transformation and invariance;
   - Gauge fixing, unphysical scalars and Faddeev-Popov ghosts;
   - Basic gauges, t'Hooft-Feynman, Landau, unitary and R_ξ;
   - Scalar sector, tadpoles;
   - Fermionic sector, masses and mixing;
   - QCD sector of the SM;
   - Feynman rules in the SM.
3. Tools for precision calculations:
   - N-loop diagrams and N-point functions;
   - Feynman parameterization; dimensional regularization;
   - Passarino–Veltman functions and reduction;
   - Infrared and ultraviolet divergences;
   - Special Passarino–Veltman functions.

4. Towards precision predictions for experimental observables:
   - Calculation of simplest QED diadrams;
     - bosonic self-energy;
     - fermionic self-energy;
     - QED vertex function;
   - Massless World, QED corrections for decay rates;
     - Phase space in n dimensions;
     - QED vertex function;
     - Bremsstrahlung in n dimensions.
5. One-loop diagrams and amplitudes:
   - More self-energies and their properties;
     - Two ways of calculation of decay rates;
     - Dispersion relation for photonic self-energy (vacuum polarization);
   - More vertices and their properties;
   - Renormalization for pedestrians: OMS-scheme;
   - Parameters Δr and $\delta\rho$;
   - Examples of finite one-loop amplitudes;
   - A short review of higher order corrections.
6. Precision calculations for LEP1/SLC:
   - Status of theoretical predictions;
   - Status of experimental data;
   - Indirect limits on Higgs boson mass;
   - Comparison of theory with experiment;
   - Future of precision high energy physics;
   - Our group and plans.

Acknowledgements:

Transparencies, to a large extent, are based on the book:
D.Bardin and G.Passarino **"The Standard Model in the Making"**,
Oxford University Press, August 1999.
I am grateful to Penka Christova, Lida Kalinovskaya and Gizo Nanava
for a critical reading and checking of transparencies.

Why these lectures?
Objectivities:
During several last years a new discipline was born Precision High-Energy Physics – **PHEP**, both in *experiment* and *theory*.
Experiment:

- *Past*: Z resonance physics at LEP1 and SLAC, unprecedented statistics, per mill level precision of measurements;
- *Present*: unexpected PHEP at LEP2!
- Bright *future* for PHEP at the nearest colliders:
  - TEVATRON, also approaches PHEP standards;
  - LHC also expects to be a typical PHEP facility;
  - LINEAR COLLIDER (LC), e.g. in GigaZ phase of LC (Z resonance mode) 100 times richer than at LEP1 statistics.

Theory:
Success of the SM in description of LEP1/SLC data. The SM finally strenghthened itself as QFT capable for precision calculation in HEP.
This status of the SM was achieved during nearly 40 year's heroic efforts of a large community of theorists' tracing back to pioneering papers by S. L. Glashow, S. Weinberg and A. Salam in the beginning of 60's, and finally recognized by the decision to award the 1999 Nobel Prize in Physics to G. t'Hooft and M. Veltman "for elucidating of quantum structure of electroweak interactions in physics", and for "having placed this theory on a firmer mathematical foundation".
Subjectivities:

- worked for about 20 years in the field of PHEP; recognize common work with Tord Riemann, started in 1983 in BLTP;
- was deeply involved in the LEP1/SLC analysis within the framework of the `ZFITTER` project and CERN Workshops (at least four) dedicated to precision calculations for experiments at LEP;
- Last, but not least the book written together with Giampiero Passarino.

Why **such** lectures?
such - based on our book, biased towards calculations.
Objectivities:

- book – we tried to show how the SM works for precision calculations of observables in e^+e^- annihilation;
- precision calculations consume a lot of mathematics;
- creation of *SCHOONSCHIP* was specially mentioned in the decision to award the 1999 Nobel Prize to Prof. M. Veltman;
- nowadays, all the cumbersome diagrammatic calculations are done with algebraic computer systems;
- lectures – an introduction to a demonstration of our site **brg.jinr.ru**, where we begin collecting everything what was done by our group;

Subjectivities:

- it is our way of understanding physics by means of calculations; when working on the book, we liked to say:
  We do not prove Ward identities – we compute them.
- lectures follow the same approach;
- first five lectures are self-contained and may be studied separately;
- lectures are appearing in CERN-JINR 1999 School Yellow Report, CERN 2000-07, 27 June 2000.

Remarks:

- Lectures are not a *simple* extraction from the book.
  I see them as *introductory* and *complimentary* to the book.
- Both in the book and in the lectures the Pauli metrics is used, i.e. for an on-mass-shell momentum one has: $p^2 = -M^2$.

  Veltman → Passarino

  Bilenky → Bardin

Quantum Fields of the SM and their Properties

Three generation of fermions or matter fields:

$$f = \begin{cases} \begin{pmatrix} \nu \\ l \end{pmatrix} = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix} \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \\ \\ \begin{pmatrix} U \\ D \end{pmatrix} = \begin{pmatrix} u \\ d \end{pmatrix} \quad \begin{pmatrix} c \\ s \end{pmatrix} \quad \begin{pmatrix} t \\ b \end{pmatrix} \end{cases}$$

They possess masses, m_f, charges, Q_f (in units of positron charge), and third projection of weak isospin, $I_f^{(3)}$:

$$m_f, \quad Q_f = \begin{pmatrix} \nu & l & U & D \\ \\ 0 & -1 & +\frac{2}{3} & -\frac{1}{3} \end{pmatrix}, \quad I_f^{(3)} = \begin{pmatrix} \nu & l & U & D \\ \\ +\frac{1}{2} & -\frac{1}{2} & +\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Gauge fields:

Vector bosons	Unphysical scalars	Faddev-Popov ghosts
A		Y^A
$Z\ (M_Z)$	ϕ^0	Y^Z
$W^\pm (M_W)$	$\phi^\pm$	$X^\pm$

[possess physical charges and unphysical masses and unphysical charges]

Higgs field:

$H\ (M_H)$ [scalar, neutral, massive]

Gluon:

g [vector, neutral, massless; possesses strong interaction]

Equations of motion and Lagrangians

Basic fields: scalar neutral and charged: $\phi^0(x)\,, \quad \phi^\pm(x)$

spinor: $\psi(x)\,, \quad \overline{\psi}(x)$

electromagnetic: $A_\alpha(x)$

vector massive, neutral and charged: $Z_\alpha(x)\,, \quad W^\pm_\alpha(x)$

satisfy *equation of motion*, free, or with *sources*.

Equations of motions for free fields:

Klein-Gordon: $\left(\Box - M^2\right)\phi^0(x) = 0, \qquad \Box = \partial_\mu\partial_\mu$

$\partial_\mu\phi^+(x)\,\partial_\mu\phi^-(x) - M^2\phi^+(x)\,\phi^-(x) = 0$

Dirac: $(\not\partial + m)\,\psi(x) = 0$

Maxwell: $\partial_\mu F_{\mu\nu} = 0, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Proca: $\partial_\mu F_{\mu\nu} - M_0^2 Z_\nu = 0, \qquad F_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$

Relation between a Lagrangian $\mathcal{L}$ and equation of motion

Euler–Lagrange equation:

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\alpha\frac{\partial\mathcal{L}}{\partial\left(\partial_\alpha\varphi\right)} = 0$$

where all fields φ and all their derivatives $\partial_\alpha\varphi$ ($\varphi = \phi^0,\ \phi^\pm,\ \psi,\ \overline{\psi},\ A_\alpha\,,\ Z_\alpha$) etc., should be considered as independent variables *at variation*.

Example of neutral vector field

$$\mathcal{L} \;=\; -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} - \frac{1}{2}M_0^2 Z_\mu Z_\mu$$

and computing derivatives

$$\frac{\partial\mathcal{L}}{\partial Z_\nu} = -M_0^2 Z_\nu\,, \qquad \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu Z_\nu\right)} = -F_{\mu\nu}$$

$$\frac{\partial\mathcal{L}}{\partial Z_\nu} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu Z_\nu\right)} = \partial_\mu F_{\mu\nu} - M_0^2 Z_\nu = 0$$

Note 1/2 in the Lagrangian for neutral fields contrary to the $\mathcal{L}$ for charged fields. $W^\pm_\alpha$ are independent and 2 doesn't arise at variation.

Example of QED. The Lagrangian with interaction

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \overline{\psi} \left(\not{\partial} - ieQ_f \not{A} + m \right) \psi, \qquad \not{A} = A_\mu \gamma_\mu$$

Compute derivatives over all independent fields and derivatives

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial A_\nu} &= \overline{\psi} ieQ_f \gamma_\nu \psi, & \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu A_\nu \right)} &= -F_{\mu\nu} \\
\frac{\partial \mathcal{L}}{\partial \overline{\psi}} &= -\left(\not{\partial} - ieQ_f \not{A} + m \right) \psi, & \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \overline{\psi} \right)} &= 0 \\
\frac{\partial \mathcal{L}}{\partial \psi} &= -\overline{\psi} \left(-ieQ_f \not{A} + m \right), & \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \psi \right)} &= -\overline{\psi} \gamma_\mu
\end{aligned}$$

and the system of **three** Euler–Lagrange equations is

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu A_\nu \right)} &= \overline{\psi} ieQ_f \gamma_\nu \psi + \partial_\mu F_{\mu\nu} = 0 \\
\frac{\partial \mathcal{L}}{\partial \overline{\psi}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \overline{\psi} \right)} &= -\left(\not{\partial} - ieQ_f \not{A} + m \right) \psi = 0 \\
\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \psi \right)} &= -\overline{\psi} \left(-ieQ_f \not{A} + m \right) + \partial_\mu \overline{\psi} \gamma_\mu = 0
\end{aligned}$$

or equivalently

$$\begin{aligned}
\partial_\mu F_{\mu\nu} &= -ieQ_f \overline{\psi} \gamma_\mu \psi \\
\left(\not{\partial} + m \right) \psi &= ieQ_f \not{A} \psi \\
\overline{\psi} \left(\not{\partial} - m \right) &= -ieQ_f \overline{\psi} \not{A}
\end{aligned}$$

that is Maxwell equation with current (source) and two equations for ψ and Dirac-conjugated field $\overline{\psi}$ with sources.

In QFT language one says that sources *emit/absorb* e^+e^--pairs, γe^- and γe^+, correspondingly.

$\mathcal{S}$ matrix and amplitude of a process

A process

$$\begin{aligned} p_1 + p_2 &\rightarrow p_1' + p_2' + \cdots \\ P &= p_1 + p_2 \,, \qquad\qquad \text{initial momentum} \\ P' &= p_1' + p_2' + \cdots , \quad \text{final momentum} \end{aligned}$$

(p_i denotes simultaneously a *particle* and its *4-momentum*)

is characterized in QFT by a matrix element:

$$\langle f|\mathcal{S} - 1|i\rangle = \langle f|\mathcal{R}|i\rangle \,(2\pi)^4 \,\delta\,(P' - P)$$

where $\mathcal{S}$ matrix

$$\mathcal{S} \;=\; T\left\{\exp\left[i \int \mathcal{L}_{\mathrm{I}}\,(x)\, d^4x\right]\right\}$$

is derived from an *interaction* Lagrangian, $\mathcal{L}_{\mathrm{I}}$, with the aid of a *time-ordering* operation, T.

$\mathcal{L}_{\mathrm{I}} \propto$ *coupling constant*, that is usually small and a *perturbation expansion* for a process amplitude is being developped.

Quantum fields which the Lagrangian is made of, may **act** on *initial* and *final* states $|i\rangle$ and $\langle f|$, giving rise to plane waves describing *in* and *out* particles, or **contract** with each other, giving rise to propagators.

Feynman rules for external lines, vertices and propagators offer very transparent way of construction of process amplitudes, *order-by-order* in perturbation theory.

Typical Feynman rule for an external line

$\vec{p}$

$$\rightarrow \quad \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2p_0}} \times \left[1, \bar{u}\,(p)\,, \epsilon_\mu\,(p)\,, e_\mu\,(p)\,, \ldots\right]$$

Cross-sections and decay rates

Total transition probability (in whole space-time)

$$dW_{fi} = \mid \langle f|\mathcal{R}|i\rangle \mid^2 (2\pi)^8 \delta(P' - P) \frac{1}{(2\pi)^4} \int e^{i(P'-P)x} d^4x d^3p'_1 d^3p'_2 \cdots$$

Transition probability in unit of time per unit of volume

$$dw_{fi} = \lim_{V,T\to\infty} \frac{dW_{fi}}{VT} = \mid \langle f|\mathcal{R}|i\rangle \mid^2 (2\pi)^4 \delta(P' - P) d^3p'_1 d^3p'_2 \cdots$$

Differential cross-section

$$d\sigma_{fi} = \frac{dw_{fi}}{j}$$

where j is the *initial flux*

$$j = \rho_1\rho_2 \frac{\sqrt{(p_1p_2)^2 - m_1^2m_2^2}}{(p_1)_0 (p_2)_0}$$

with

$$\rho_i = \frac{1}{(2\pi)^3} \quad \text{and} \quad N_{p_k} = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2(p_k)_0}}$$

the differential cross-section becomes

$$d\sigma_{fi} = \frac{1}{4\sqrt{(p_1p_2)^2 - m_1^2m_2^2}} \mid \mathcal{M}_{fi} \mid^2 d\Phi_n$$

where

$$d\Phi_n = (2\pi)^4 \prod_{k=1}^{n} \frac{d^3p'_k}{(2\pi)^3 2(p'_k)_0} \delta\left(\sum_{j=1}^{n} p'_j - P\right)$$

Process matrix element squared

$$N_{p_1}^2 N_{p_2}^2 \prod_{k=1}^{n} N_{p'_k}^2 \mid \mathcal{M}_{fi} \mid^2 = \overline{\sum_{\text{spins}}} \mid \langle f|\mathcal{R}|i\rangle \mid^2$$

is defined with separated out N_{p_k} and should be understood as *averaged* over initial and *summed* over final spin degrees of freedom.

For the decay rate of the process

$$P \ \to \ p'_1 + p'_2 + \cdots$$

one has

$$d\Gamma_{fi} \ = \ \frac{dw_{fi}}{\rho}$$

where $\rho = \dfrac{1}{(2\pi)^3}$ is initial density.

Similarly one gets

$$d\Gamma_{fi} \ = \ \frac{1}{2P_0} \mid \mathcal{M}_{fi} \mid^2 d\Phi_n$$

Note a difference with PDG convention: $(2\pi)^4$ is shifted the phase space. This is convenient for calculation in n dimensions.

Input parameters in the Standard Model

Theory			parameters			N_p
Conventional QED	$\to$	e	m_e			2
Extended QED	$\to$	e	m_e	m_μ	m_τ	
			m_u	m_c	m_t	
			m_d	m_s	m_b	10
EW Standard Model	$\to$	$+$	M_W	M_Z	M_H	
			4 mixing angles			17
Conventional SM	$\to$	$+\ \alpha_S$				18
Extended SM	$\to$	$+$	m_{ν_e}	m_{ν_μ}	m_{ν_τ}	
			4 mixing angles			25

Number of parameters is large, however, this is trivial consequence of large number of *fundamental fields* and objective **complexity of Nature**. This number, however, is *minimal* :

- three generations is a *minimal* number, needed to have CP violation, which exists in Nature, remember

$$N_{\text{phases}} = \frac{(N_g - 1)\,(N_g - 2)}{2}\,, \qquad N_g - \text{number of generations}$$

all 9 fundamental fermions are experimentally found;

- three gauge bosons is a *minimal* number, needed to describe all EW interactions existing in Nature,

all 3 gauge bosons are experimentally found;

- fermionic mixing is unavoidable and exists in Nature both in *hardonic* and *leptonic* worlds, **CKM mixing is experimentally well measured, ν-mixing is possibly discovered;**
- *only Higgs boson is not yet found.* There are *indirect* indications.

The Standard Model **is not able** to calculate these 25 parameters (in this sence the SM *does not predict them*).

This is why people believe that some day a better theory will be discovered.

This is why people want to find some experimental indications on new physics *beyond the SM* and built new accelerators, LHC...

So far, neither experiment has found strong evidences of new physics, (*situation with the description of* **all** ν *-data has to be clarified yet*) nor theory proposed the explanation of the *whole* mass spectrum of fundamental particles ranging

from parts of eV for lightest neutrino
to 175 GeV for heaviest top quark
more than 12 orders of magnitude

Standard Model **is able** to calculate *any experimental observable* $O_i^{\rm exp}$ in terms of an Input Parameter Set (IPS). We define

$$\text{IPS} \quad \equiv \quad 25 \text{ parameters}$$

and what we do within the SM, symbolically

$$O_i^{\rm exp} \leftrightarrow O_i^{\rm theor}\,(\text{IPS})$$

Input Parameters are experimentally known with different precisions:

$$\begin{array}{rcll} m_e & = & 0.51099907 \pm 0.00000015 \text{ MeV} & \sim\ 3 \times 10^{-7} \\ M_Z & = & 91.1871 \pm 0.0021 \text{ GeV} & \sim\ 2 \times 10^{-5} \\ M_W & = & 80.394 \pm 0.042 \text{ GeV} & \sim\ 5 \times 10^{-4} \\ m_t & = & 174.3 \pm 5.1 \text{ GeV} & \sim\ 3 \times 10^{-2} \\ M_H & \leq & 215 \text{ GeV} \quad (95\%\ c.l.) \quad \text{indirectly} & \end{array}$$

Precision measurements provide *constraints* on Input Parameters. This is how one may *extract an information* on yet unknown parameters (or improve our knowledge of poorly measured ones).
This shouldn't be mixed with **prediction** in above mentioned sence. (W, Z, t, H-story).

Conventional QED.

The electron anomaly $a_e = (g_e - 2)/2$:

$$\begin{aligned} a_e^{\rm exp} &= 1159652193(10) \times 10^{-12} \\ a_e^{\rm th} &= 1159652140(27) \times 10^{-12} \qquad \text{up to fourth order } \mathcal{O}(\alpha^4) \end{aligned}$$

8 digits agreement! Can't be by chance.

Conventional EW.

The Z resonance observables measured at LEP1 (CERN) and SLC (SLAC) with

$$\text{experimental precision} \leq 10^{-3}$$

Therefore, one needs

$$\text{theoretical precision} \sim 2.5 \times 10^{-4}$$

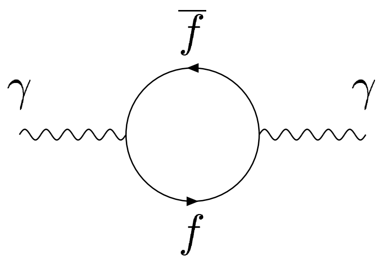
Number of free parameters in fits of Z-resonance observables

Lepton masses are known very precise, the worst

$$m_\tau = 1777.05^{+0.29}_{-0.26}\ \text{MeV} \quad < 10^{-4}$$

is infinitely precise in typical LEP1 precision scale 10^{-3}

Vacuum polarization



leads to

$$\sum_f \ln \frac{s}{m_f^2}$$

no problem for leptons.

Light quark masses should be replaced by the experimentally measured quantity $\sigma(e^+e^- \to \text{hadrons}) \quad \to \quad \alpha\left(M_Z^2\right)$.

So, we are left with 6 parameters only (**the standard SM IPS**)

$$\alpha\left(M_Z^2\right) \qquad \alpha_S\left(M_Z^2\right) \qquad m_t \qquad M_Z \qquad M_W \qquad M_H$$

And for three of them

$$\alpha_S\left(M_Z^2\right) \qquad m_t \qquad M_W$$

reach information is available from the other then LEP1/SLC measurements.
Number of free parameters is very few!

<u>More on coupling constants, typical scales</u>

LEP typical scale: $\sqrt{s} \sim M_Z - 200$ GeV
typical EW scale: $100 \div 300$ GeV

$$\begin{aligned}
\alpha(0)L &= 0.169 && \text{up to third order in QED} && \mathcal{O}\left(\alpha^3 L^3\right)\\
& && L = \ln\frac{s}{m_e^2} - 1 \sim 23 \text{ at } s = M_Z^2 &&\\
\alpha\left(M_Z^2\right) &\equiv \alpha_Z = 1/128.9 && \text{up to second order in EW sector} && \mathcal{O}\left(\alpha_Z^2\right)\\
\alpha_S\left(M_Z^2\right) &= 0.119 \pm 0.003 && \text{up to third order in QCD sector} && \mathcal{O}\left(\alpha_S^3\right)
\end{aligned}$$

mixed corrections are also needed

$$\mathcal{O}\left(\alpha\alpha_S\right), \qquad \mathcal{O}\left(\alpha\alpha_S^2\right)$$

Since

$$\begin{aligned}
M_W &\sim 80 \text{ GeV}\\
M_Z &\sim 91 \text{ GeV}\\
m_t &\sim 175 \text{ GeV}\\
M_H &\leq 300 \text{ GeV}
\end{aligned}$$

the calculation must be exact in all these masses. Since the next after top heavy fermion b-quark has mass ~ 4.5 GeV, it is sufficient to keep first order in

$$\frac{m_b^2}{s}$$

all the other masses may be safely neglected.

LEP tandems

`ZFITTER`

D. Bardin, M. Bilenky, P. Christova, M. Jack, L. Kalinovskaya, A. Olchevski, S. Riemann, T. Riemann

`GENTLE/4fan`

D. Bardin, M. Bilenky, J. Biebel, D. Lehner, A. Leike, A. Olchevski, T. Riemann

`TOPAZO`

G. Montagna, O. Nicrosini, F. Piccinini, G. Passarino,

`WTO, ZTO`

G. Passarino

$\mathcal{KK}$MC

S. Jadach, B. Ward, Z. Wąs

`KORALW`

S. Jadach, W. Placzek, M. Skrzypek, Z. Wąs, B. F. L. Ward

QED Lagrangian

$$\mathcal{L}^0_{\mathrm{QED}} = -\frac{1}{4}\,F_{\mu\nu}F_{\mu\nu} - \frac{1}{2}\,(\mathcal{C}^A)^2 - \sum_f \overline{\psi}_f\;(\partial\!\!\!/ + m_f)\,\psi_f$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \qquad \partial\!\!\!/ = \partial_\mu\gamma_\mu\,, \qquad \mathcal{C}^A = -\frac{1}{\xi}\,\partial_\mu A_\mu$$

We use the $4\,\times\,4$ representation:

$$\gamma_j \;=\; \begin{pmatrix} O & -i\,\tau_j \\ i\,\tau_j & O \end{pmatrix}, \quad j = 1,2,3; \qquad \gamma_4 = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}$$

$$\gamma_5 \;=\; \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} O & -I \\ -I & O \end{pmatrix}; \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Properties of γ matrices

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}\,, \qquad \gamma^+_\mu = \gamma_\mu\,, \qquad \gamma^2_\mu = I$$

Pauli matrices,

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and their properties

$$\tau_i = \tau^+_i\,, \qquad \tau_i\tau^+_i = I\,, \qquad \tau_i\tau_i = I$$

$$\left[\frac{1}{2}\tau_i, \frac{1}{2}\tau_j\right] = i\varepsilon_{ijk}\frac{1}{2}\tau_k \qquad \tau_i\tau_j = \delta_{ij} + i\varepsilon_{ijk}\tau_k$$

$\frac{1}{2}\tau_i - SU(2)$ generators

$$U = \exp\left\{-i\frac{1}{2}\tau_i\lambda_i\right\}, \qquad UU^+ = I, \qquad detU = 1$$

Free-particle spinors satisfy the following relations:

$$\begin{aligned} (ip\!\!\!/ + m)\;u(p) \;&=\; 0, \qquad (-ip\!\!\!/ + m)\;v(p) = 0 \\ \overline{u}(p)\;(ip\!\!\!/ + m) \;&=\; 0, \qquad \overline{v}(p)\;(-ip\!\!\!/ + m) = 0 \end{aligned}$$

Local gauge transformation and invariance

Local gauge tansformations

$$
\begin{aligned}
\psi'_f(x) &= e^{-ieQ_f\lambda(x)}\psi_f(x) \\
\overline{\psi}'_f(x) &= \overline{\psi}_f(x)e^{ieQ_f\lambda(x)} \\
A'_\mu(x) &= A_\mu(x) - \partial_\mu\lambda(x)
\end{aligned}
$$

The full Lagrangian $\mathcal{L}^{\mathrm{I}}_{\mathrm{QED}}$ will be invariant under local gauge tansformations if we replace ∂_μ by covariant derivative

$$\partial_\mu \to D_\mu = \partial_\mu - ieQ_f A_\mu$$

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} - \frac{1}{2}\left(\mathcal{C}^A\right)^2 - \sum_f \overline{\psi}_f\,\left(\not\partial - ieQ_f\not A + m_f\right)\psi_f$$

(where e is positive and $e^2 = 4\pi\alpha$, i.e. *positron* charge, and Q_f is (*fraction of charge*)$\times 2I_f^{(3)}$: $Q_l = -1$, $Q_u = +2/3$ and $Q_d = -1/3$)

Proof of gauge invariance

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu A_\nu - \partial_\mu\partial_\nu\lambda(x) - \partial_\nu A_\mu + \partial_\mu\partial_\nu\lambda(x) = F_{\mu\nu}$$

$$m_f\overline{\psi}'_f(x)\psi'_f(x) = m_f\overline{\psi}_f(x)\psi_f(x)$$

$$\overline{\psi}'_f(x)\left(\partial_\mu - ieQ_f A'_\mu\right)\psi'_f(x) =$$

$$\overline{\psi}_f(x)e^{ieQ_f\lambda(x)}\Big[\partial_\mu - ieQ_f\partial_\mu\lambda(x) - ieQ_f\left(A_\mu(x) - \partial_\mu\lambda(x)\right)\Big]e^{-ieQ_f\lambda(x)}\psi_f(x)$$

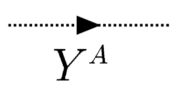
$$= \overline{\psi}_f(x)\left(\partial_\mu - ieQ_f A_\mu\right)\psi_f(x)$$

Subjecting $\mathcal{C}^A$ to our gauge transformation

$$-\frac{1}{\xi}\partial_\mu A'_\mu = -\frac{1}{\xi}\partial_\mu A_\mu + \frac{1}{\xi}\partial_\mu\partial_\mu\lambda(x), \qquad \partial_\mu\partial_\mu = \Box, \qquad \Box\lambda(x) = 0$$

we discover a massless, non-interacting ghost field $\lambda(x) \equiv Y^A(x)$ with propagator

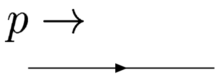
$$\frac{1}{\xi}\Box$$



$$\frac{\xi}{p^2}$$

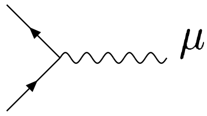
Feynman rules of QED:

Could be easily **guessed** looking at the Lagrangian



$$\frac{1}{(2\pi)^4 i} \frac{1}{i\not{p} + m_f} = \frac{1}{(2\pi)^4 i} \frac{-i\not{p} + m_f}{p^2 + m_f^2 - i\epsilon}$$

$$\frac{1}{(2\pi)^4 i} \frac{1}{p^2 + i\epsilon} \left[\delta_{\mu\nu} + \left(\xi^2 - 1\right) \frac{p_\mu p_\nu}{p^2}\right]$$



$$(2\pi)^4 i \; ieQ_f \, \gamma_\mu$$

Note appearance of ξ-dependent term in photonic propagator, consequence of gauge fixing.

Photon propagators in three gauges:

- General R_ξ, propagator as given above
- Feynman gauge, $\xi = 1$

$$\frac{1}{(2\pi)^4 i} \frac{\delta_{\mu\nu}}{p^2 + i\epsilon}$$

- Landau gauge, $\xi = 0$

$$\frac{1}{(2\pi)^4 i} \frac{1}{p^2 + i\epsilon} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)$$

Usually in QED one uses the Feynman gauge. It is well known that the ξ-dependence cancels in the $\mathcal{S}$-matrix for a given physical process.

As example consider any $e^+e^- \rightarrow \gamma^*$ sub-process. The corresponding $\mathcal{S}$-matrix element in the R_ξ-gauge will have an additional term

$$-\left(\xi^2 - 1\right) \bar{v}\left(p_+\right) \left(\not{p}_+ + \not{p}_-\right) u\left(p_-\right),$$

which is zero for on-mass-shell fermions by virtue of Dirac equation.

The extra term, proportional to $\xi^2 - 1$, may be omitted.

Standard Model (SM) Lagrangian in the R_ξ gauge

Reminder: SM fields content

- triplet of vector bosons B^a_μ, a singlet B^0_μ
- a complex scalar field K, (minimal – one doublet of complex fields)
- fermion families
- Faddeev–Popov ghost-fields $X^\pm, Y^Z, Y^A$

The total Lagrangian is the sum of various pieces.

The first piece is the standard Yang–Mills Lagrangian

$$\begin{aligned}\mathcal{L}_{\mathrm{YM}} &= -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - \frac{1}{4}F^0_{\mu\nu}F^0_{\mu\nu}\\ F^a_{\mu\nu} &= \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + g\varepsilon_{abc}B^b_\mu B^c_\nu\\ F^0_{\mu\nu} &= \partial_\mu B^0_\nu - \partial_\nu B^0_\mu\end{aligned}$$

Yang–Mills Lagrangian is invariant under local $SU(2)\times U(1)$ gauge transformations if in the corresponding free field Lagrangian one replaces:

$$\partial_\mu \to D_\mu = \partial_\mu - \frac{i}{2}gB^a_\mu\tau^a - \frac{i}{2}gg_iB^0_\mu$$

g – $SU(2)$ bare coupling constant and g_i – an arbitrary hypercharge.

The physical fields Z and A are related to B^3_μ and B^0_μ by

$$\begin{pmatrix} Z \\ A \end{pmatrix} = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix}\begin{pmatrix} B^3 \\ B^0 \end{pmatrix}$$

where $s_\theta(c_\theta)$ denote the sine and cosine of the weak mixing angle.

Second piece: the *minimal* Higgs sector (scalar) Lagrangian

$$\mathcal{L}_{\mathrm{S}} = -\left(D_\mu K\right)^+ D_\mu K - \mu^2 K^+K - \frac{1}{2}\lambda\left(K^+K\right)^2$$

where $\lambda > 0$ and symmetry breaking requires $\mu^2 < 0$.

The scalar field in the minimal realization of the SM is

$$K = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi \\ \sqrt{2} i\phi^- \end{pmatrix}, \qquad \chi = H + \langle v \rangle + i\phi^0$$

Four scalar fields: $\phi^\pm$, ϕ^0 and H; H – physical Higgs boson field, $\langle v \rangle$ – vacuum expectation value (v.e.v.).

The covariant derivative for the scalar field in $SU(2) \otimes U(1)$

$$D_\mu K = \left(\partial_\mu - \frac{i}{2} g B^a_\mu \tau^a - \frac{i}{2} g g_1 B^0_\mu \right) K$$

The hypercharge g_1 to be fixed below!

The scalar field can be rewritten as

$$K = \frac{1}{\sqrt{2}} \left(H + \langle v \rangle + i\phi^a \tau^a \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and the covariant derivative as

$$\begin{aligned} D_\mu K &= \frac{1}{\sqrt{2}} \left(\partial_\mu - \frac{i}{2} g B^a_\mu \tau^a - \frac{i}{2} g g_1 B^0_\mu \right) \left(H + \langle v \rangle + i\phi^b \tau^b \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \left\{ \partial_\mu H - \frac{i}{2} g g_1 B^0_\mu \left(H + \langle v \rangle \right) + \frac{1}{2} g B^a_\mu \phi^a \right. \\ &\left. + i \left[\partial_\mu \phi^a - \frac{1}{2} g B^a_\mu \left(H + \langle v \rangle \right) - \frac{i}{2} g g_1 B^0_\mu \phi^a + \frac{1}{2} g \varepsilon_{cba} B^c_\mu \phi^b \right] \tau^a \right\} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

Similarly

$$\begin{aligned} (D_\mu K)^+ &= (1,0) \frac{1}{\sqrt{2}} \left\{ \partial_\mu H + \frac{i}{2} g g_1 B^0_\mu \left(H + \langle v \rangle \right) + \frac{1}{2} g B^a_\mu \phi^a \right. \\ &\left. - i \left[\partial_\mu \phi^a - \frac{1}{2} g B^a_\mu \left(H + \langle v \rangle \right) + \frac{i}{2} g g_1 B^0_\mu \phi^a + \frac{1}{2} g \varepsilon_{cba} B^c_\mu \phi^b \right] \tau^a \right\} \end{aligned}$$

Consider the product

$$- \left(D_\mu K \right)^+ D_\mu K$$

first term of $\mathcal{L}_{\mathrm{S}}$, containing 81 terms!

Introduce physical fields

$$W^{\pm}_\mu = \frac{1}{\sqrt{2}}\left(B^1_\mu \mp iB^2_\mu\right), \qquad \phi^{\pm} = \frac{1}{\sqrt{2}}\left(\phi^1 \mp i\phi^2\right), \qquad \phi^0 \equiv \phi^3$$

$$Z_\mu = c_\theta B^3_\mu - s_\theta B^0_\mu, \qquad A_\mu = s_\theta B^3_\mu + c_\theta B^0_\mu$$

and collect first of all terms with $\langle v\rangle^2$

$$-(1,0)\frac{1}{\sqrt{2}}\left\{\frac{i}{2}gg_1B^0_\mu\langle v\rangle + i\frac{1}{2}gB^b_\mu\langle v\rangle\tau^b\right\}$$

$$\times\frac{1}{\sqrt{2}}\left\{-\frac{i}{2}gg_1B^0_\mu\langle v\rangle - i\frac{1}{2}gB^c_\mu\langle v\rangle\tau^c\right\}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= -\frac{g^2\langle v\rangle^2}{8}(1,0)\left(g_1B^0_\mu + B^c_\mu\tau^c\right)\left(g_1B^0_\mu + B^b_\mu\tau^b\right)\begin{pmatrix}1\\0\end{pmatrix} \to$$

$$(1,0)\,B^a_\mu\tau^a\begin{pmatrix}1\\0\end{pmatrix} = B^3_\mu$$

$$(1,0)\,B^c_\mu\tau^cB^b_\mu\tau^b\begin{pmatrix}1\\0\end{pmatrix} = B^a_\mu B^a_\mu$$

$$\to -\frac{g^2\langle v\rangle^2}{8}\left(g_1^2B^0_\mu B^0_\mu + 2g_1B^0_\mu B^3_\mu + B^a_\mu B^a_\mu\right)$$

$$= -\frac{g^2\langle v\rangle^2}{8}\left[\left(g_1B^0_\mu + B^3_\mu\right)^2 + B^1_\mu B^1_\mu + B^2_\mu B^2_\mu\right]$$

and if one chooses $g_1 = -s_\theta/c_\theta$ then

$$= -\frac{g^2\langle v\rangle^2}{8}\left[\frac{1}{c^2_\theta}(Z_\mu)^2 + 2W^+_\mu W^-_\mu\right] = -\frac{1}{2}M^2_0(Z_\mu)^2 - M^2W^+_\mu W^-_\mu$$

Higgs mechanism generates masses:

M – bare mass of W boson, $M = \dfrac{g\langle v\rangle}{2}$

M_0 – bare mass of Z boson, $M_0 = \dfrac{g\langle v\rangle}{2c_\theta}$

Or equivalently: $c_\theta = \dfrac{M}{M_0}$ and $\langle v\rangle = 2\dfrac{M}{g}$.

Continue our consideration of the product $-\left(D_\mu K\right)^+ D_\mu K$. Substitute $\langle v\rangle$ and look at all terms **without** interaction constant g

$$-\left(1,0\right)\frac{1}{\sqrt{2}}\left[\partial_\mu H+iMg_1B_\mu^0-i\left(\partial_\mu\phi^c-MB_\mu^c\right)\tau^c\right]$$

$$\times\frac{1}{\sqrt{2}}\left[\partial_\mu H-iMg_1B_\mu^0+i\left(\partial_\mu\phi^b-MB_\mu^b\right)\tau^b\right]\begin{pmatrix}1\\0\end{pmatrix}\rightarrow$$

Omitting **legal** kinetic terms like

$$-\frac{1}{2}\left(\partial_\mu H\right)^2 \qquad \text{etc.}$$

terms which were already considered (mass terms) and observing that $HB_\mu^{a,0}$ transitions cancel identically, we are left with

$$\rightarrow\frac{M}{2}\left(1,0\right)\left(g_1B_\mu^0+B_\mu^c\tau^c\right)\left(\partial_\mu\phi^b\right)\tau^b\begin{pmatrix}1\\0\end{pmatrix}$$

$$+\frac{M}{2}\left(1,0\right)\left(\partial_\mu\phi^c\right)\tau^c\left(g_1B_\mu^0+B_\mu^b\tau^b\right)\begin{pmatrix}1\\0\end{pmatrix}\rightarrow$$

Taking into account

$$\left(1,0\right)\left(B_\mu^c\tau^c\partial_\mu\phi^b\tau^b+\partial_\mu\phi^c\tau^cB_\mu^b\tau^b\right)\begin{pmatrix}1\\0\end{pmatrix}=2\delta_{bc}IB_\mu^b\partial_\mu\phi^c$$

we arrive at

$$\rightarrow M\left(g_1B_\mu^0\partial_\mu\phi^0+B_\mu^a\partial_\mu\phi^a\right)\rightarrow$$

or in terms of physical fields

$$\rightarrow M\left(\frac{1}{c_\theta}Z_\mu\partial_\mu\phi^0+W_\mu^+\partial_\mu\phi^-+W_\mu^-\partial_\mu\phi^+\right)$$

And this is **criminal** since the Lagrangian shows up $Z-\phi^0, W^\pm-\phi^\mp$ *transitions* of the zeroth order in the coupling constant and their contribution must be summed up to all orders if we want to develop perturbation theory.

To circumvent this problem we add a *gauge-fixing* piece to the Lagrangian, $\mathcal{L}_{\rm gf}$, that cancels these mixing terms.

However, it breaks the gauge invariance and we must introduce Faddeev–Popov ghost fields to compensate this breaking.

The gauge-fixing piece is (generalized R_ξ gauge)

$$\mathcal{L}_{\rm gf} = -\mathcal{C}^+\mathcal{C}^- - \frac{1}{2}\left[\left(\mathcal{C}^Z\right)^2 + \left(\mathcal{C}^A\right)^2\right]$$

where

$$\begin{aligned}
\mathcal{C}^A &= -\frac{1}{\xi_A}\partial_\mu A_\mu \\
\mathcal{C}^Z &= -\frac{1}{\xi_Z}\partial_\mu Z_\mu + \xi_Z \frac{M}{c_\theta}\phi^0 \\
\mathcal{C}^\pm &= -\frac{1}{\xi}\partial_\mu W^\pm_\mu + \xi M \phi^\pm
\end{aligned}$$

Note appearance of three different gauge parameters associated with three different vector fields: $W^\pm$, Z, A.
Consider for instance

$$\begin{aligned}
&-\frac{1}{2}\left(\mathcal{C}^Z\right)^2 = -\frac{1}{2}\left(-\frac{1}{\xi_Z}\partial_\mu Z_\mu + \xi_Z\frac{M}{c_\theta}\phi^0\right)^2 \\
&= -\frac{1}{2}\frac{1}{\xi_Z^2}\left(\partial_\mu Z_\mu\right)^2 + \frac{M}{c_\theta}\left(\partial_\mu Z_\mu\right)\phi^0 - \frac{1}{2}\left(\xi_Z\frac{M}{c_\theta}\phi^0\right)^2
\end{aligned}$$

The first and third term modify Z propagator, while the second term together with **illegal** $Z-\phi^0$ transition gives full derivative

$$\frac{M}{c_\theta}\left(Z_\mu\partial_\mu\phi^0 + \left(\partial_\mu Z_\mu\right)\phi^0\right) = \frac{M}{c_\theta}\partial_\mu\left(Z_\mu\phi^0\right)$$

that does not contribute to the Lagrangian.

In order to define the FP ghost Lagrangian we must subject $\mathcal{C}^{A,Z,\pm}$ to a gauge transformation. This is, in principle, similar to what we did in QED. Contrary to QED, there are ghost interactions in the SM.

In the R_ξ gauge

$$\mathcal{L}_{\rm YM} - (D_\mu K)^+ D_\mu K - \mathcal{C}^+\mathcal{C}^- - \frac{1}{2}(\mathcal{C}^Z)^2 - \frac{1}{2}(\mathcal{C}^A)^2 = \mathcal{L}_{\rm prop} + \mathcal{L}^{\rm bos,I}$$

The quadratic part of the Lagrangian, $\mathcal{L}_{\rm prop}$,

$$\begin{aligned}
\mathcal{L}_{\rm prop} \;=\; & -\partial_\mu W^+_\nu \partial_\mu W^-_\nu + \left(1 - \frac{1}{\xi^2}\right) \partial_\mu W^+_\mu \partial_\nu W^-_\nu \\
& -\frac{1}{2}\partial_\mu Z_\nu \partial_\mu Z_\nu + \frac{1}{2}\left(1 - \frac{1}{\xi_Z^2}\right) (\partial_\mu Z_\mu)^2 \\
& -\frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu + \frac{1}{2}\left(1 - \frac{1}{\xi_A^2}\right) (\partial_\mu A_\mu)^2 \\
& -\frac{1}{2}\partial_\mu H \partial_\mu H - \partial_\mu \phi^+ \partial_\mu \phi^- - \frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 \\
& -M^2 W^+_\mu W^-_\mu - \frac{1}{2}\frac{M^2}{c_\theta^2} Z_\mu Z_\mu \\
& -\xi^2 M^2 \phi^+ \phi^- - \frac{1}{2}\xi_Z^2 \frac{M^2}{c_\theta^2} \phi^0 \phi^0 - \frac{1}{2} M_H H^2
\end{aligned}$$

The scalar field propagators are trivially guessed from $\mathcal{L}_{\rm prop}$

$$-\partial_\mu \phi^+ \partial_\mu \phi^- - \xi^2 M^2 \phi^+ \phi^- \;\to\; \frac{1}{p^2 + \xi^2 M^2} \qquad \text{etc.}$$

The *rule of correspondence* for vector fields is more complicated

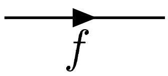
$$-\frac{1}{2}\partial_\mu Z_\nu \partial_\mu Z_\nu + \frac{1}{2}\left(1 - \frac{1}{\xi^2}\right)(\partial_\mu Z_\mu)^2 + \frac{1}{2} M_0^2 Z_\mu Z_\mu$$

$$\to \; \frac{\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}}{p^2 + M_0^2} + \frac{\frac{p_\mu p_\nu}{p^2}}{\frac{1}{\xi^2} p^2 + M_0^2}$$

It is proved in standard textbooks on QFT.

Feynman rules for propagators (full collection)

Propagator of a fermion, f



$$\frac{-i\not{p}+m_f}{p^2+m_f^2}$$

Vector boson propagators

A 

$$\frac{1}{p^2}\left\{\delta_{\mu\nu}+\left(\xi_A^2-1\right)\frac{p_\mu p_\nu}{p^2}\right\}$$

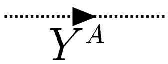
Z 

$$\frac{1}{p^2+M^2}\left\{\delta_{\mu\nu}+\left(\xi_Z^2-1\right)\frac{p_\mu p_\nu}{p^2+\xi_Z^2 M^2}\right\}$$

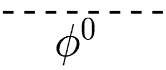
$W^\pm$ 

$$\frac{1}{p^2+M^2}\left\{\delta_{\mu\nu}+\left(\xi^2-1\right)\frac{p_\mu p_\nu}{p^2+\xi^2 M^2}\right\}$$

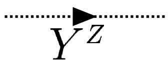
Propagators of unphysical fields



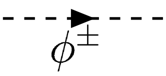
$$\frac{\xi_A}{p^2}$$



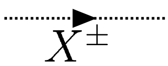
$$\frac{1}{p^2+\xi_Z^2\dfrac{M^2}{c_\theta^2}}$$



$$\frac{\xi_Z}{p^2+\xi_Z^2\dfrac{M^2}{c_\theta^2}}$$

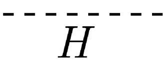


$$\frac{1}{p^2+\xi^2 M^2}$$



$$\frac{\xi}{p^2+\xi^2 M^2}$$

Propagator of physical scalar field, H-boson



$$\frac{1}{p^2+M_H^2}$$

Every propagator should be multiplied by factor $\dfrac{1}{(2\pi)^4\, i}$.

More about propagators in different gauges

Three forms for $W(Z)$ boson propagator (for Z $[\xi \to \xi_Z]$)

$$W^{\pm} \to \frac{1}{p^2+M^2}\left\{\delta_{\mu\nu} + \left(\xi^2-1\right)\frac{p_\mu p_\nu}{p^2+\xi^2 M^2}\right\} \qquad R_\xi\text{-gauge}$$

$$= \frac{1}{p^2+M^2}\left(\delta_{\mu\nu}+\frac{p_\mu p_\nu}{M^2}\right) - \frac{p_\mu p_\nu}{M^2\left(p^2+\xi^2 M^2\right)}$$

$$= \frac{1}{p^2+M^2}\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right) + \frac{\xi^2}{p^2+\xi^2 M^2}\frac{p_\mu p_\nu}{p^2}$$

$$= \frac{\delta_{\mu\nu}}{p^2+M^2} \qquad \text{for} \quad \xi = 1 \qquad \text{t'Hooft-Feynman gauge}$$

$$= \frac{1}{p^2+M^2}\left(\delta_{\mu\nu}+\frac{p_\mu p_\nu}{M^2}\right) \qquad \text{for} \quad \xi = \infty \qquad \text{Unitary gauge}$$

$$= \frac{1}{p^2+M^2}\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right) \qquad \text{for} \quad \xi = 0 \qquad \text{Landau gauge}$$

Not all gauges are possible for photon propagator

$$A \to \frac{1}{p^2}\left\{\delta_{\mu\nu} + \left(\xi_A^2-1\right)\frac{p_\mu p_\nu}{p^2}\right\} \qquad R_\xi\text{-gauge}$$

$$= \frac{\delta_{\mu\nu}}{p^2} \qquad \text{for} \quad \xi_A = 1 \qquad \text{Feynman gauge}$$

$$= \frac{1}{p^2}\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right) \qquad \text{for} \quad \xi_A = 0 \qquad \text{Landau gauge}$$

The physical gauge is recovered in the limit $\xi_A \to 1$ and $\xi_Z, \xi \to \infty$.
Therefore, it is *a mixture* of the Unitary and Feynman gauges.

Standard Model Lagrangian, Bosonic Sector

$$
\begin{aligned}
\mathcal{L}^{\rm bos,I} \;=\; & -igc_\theta\left\{\partial_\nu Z_\mu W_\mu^{[+}W_\nu^{-]} - Z_\nu W_\mu^{[+}\partial_\nu W_\mu^{-]} + Z_\mu W_\nu^{[+}\partial_\nu W_\mu^{-]}\right\} \\
& -igs_\theta\left\{\partial_\nu A_\mu W_\mu^{[+}W_\nu^{-]} - A_\nu W_\mu^{[+}\partial_\nu W_\mu^{-]} + A_\mu W_\nu^{[+}\partial_\nu W_\mu^{-]}\right\} \\
& +\frac{1}{2}g^2\left\{\left(W_\mu^+W_\nu^-\right)^2 - \left(W_\mu^+W_\mu^-\right)^2\right\} \\
& +g^2c_\theta^2\left\{Z_\mu Z_\nu W_\mu^+W_\nu^- - Z_\mu Z_\mu W_\nu^+W_\nu^-\right\} \\
& +g^2s_\theta^2\left\{A_\mu A_\nu W_\mu^+W_\nu^- - A_\mu A_\mu W_\nu^+W_\nu^-\right\} \\
& +g^2s_\theta c_\theta\left\{A_\mu Z_\nu W_\mu^{[+}W_\nu^{-]} - 2A_\mu Z_\mu W_\nu^+W_\nu^-\right\} \\
& -gMH\left\{W_\mu^+W_\nu^- + \frac{1}{2c_\theta^2}Z_\mu Z_\mu\right\} \\
& -\frac{i}{2}g\left\{W_\mu^+\left(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0\right) - W_\mu^-\left(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0\right)\right\} \\
& +\frac{1}{2}g\left\{W_\mu^+\left(H\partial_\mu\phi^- - \phi^-\partial_\mu H\right) - W_\mu^-\left(H\partial_\mu\phi^+ - \phi^+\partial_\mu H\right)\right\} \\
& +\frac{1}{2}\frac{g}{c_\theta}Z_\mu\left(H\partial_\mu\phi^0 - \phi^0\partial_\mu H\right) \\
& +ig\left(s_\theta A_\mu - \frac{s_\theta^2}{c_\theta}Z_\mu\right)MW_\mu^{[+}\phi^{-]} \\
& +ig\left(s_\theta A_\mu + \frac{c_\theta^2 - s_\theta^2}{c_\theta}Z_\mu\right)\left(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+\right) \\
& -\frac{1}{4}g^2W_\mu^+W_\mu^-\left(HH + \phi^0\phi^0 + 2\phi^+\phi^-\right) \\
& -\frac{1}{8}\frac{g^2}{c_\theta^2}Z_\mu Z_\mu\left\{HH + \phi^0\phi^0 + 2\left(c_\theta^2 - s_\theta^2\right)^2\phi^+\phi^-\right\} \\
& -\frac{1}{2}g^2\frac{s_\theta^2}{c_\theta}Z_\mu\phi^0W_\mu^{[+}\phi^{-]} - \frac{i}{2}g^2\frac{s_\theta^2}{c_\theta}Z_\mu HW_\mu^{[+}\phi^{-]} + \frac{1}{2}g^2s_\theta A_\mu\phi^0W_\mu^{[+}\phi^{-]} \\
+ \; & \frac{i}{2}g^2s_\theta A_\mu HW_\mu^{[+}\phi^{-]} - g^2\frac{s_\theta}{c_\theta}\left(c_\theta^2 - s_\theta^2\right)Z_\mu A_\mu\phi^+\phi^- - g^2s_\theta^2A_\mu A_\mu\phi^+\phi^-
\end{aligned}
$$

where the anti-symmetrized combination

$$A^{[+}B^{-]} = A^+B^- - A^-B^+$$

Standard Model Lagrangian, FP Ghost Sector

In order to define the FP ghost Lagrangian we must subject $\mathcal{C}^a$ to a gauge transformation:

$$\begin{aligned}
B^a_\mu &\to B^a_\mu + g\varepsilon_{abc}\Lambda^b B^c_\mu - \partial_\mu\Lambda^a, \qquad B^0_\mu \to B^0_\mu - \partial_\mu\Lambda^0, \\
K &\to \left(1 - \frac{i}{2}g\Lambda^a\tau^a - \frac{i}{2}gg_1\Lambda^0\right)K, \qquad \text{with} \quad g_1 = -\frac{s_\theta}{c_\theta} \\
H \pm i\phi^0 &\to H \pm i\phi^0 \mp \frac{i}{2}g\left[\left(\Lambda^3 + g_1\Lambda^0\right)\left(H + 2\frac{M}{g} \pm i\phi^0\right) \pm 2i\Lambda^\pm\phi^\mp\right] \\
\phi^0 &\to \phi^0 - \frac{1}{2}g\left(\Lambda^3 + g_1\Lambda^0\right)\left(H + 2\frac{M}{g}\right) + \frac{i}{2}g\left(\Lambda^-\phi^+ - \Lambda^+\phi^-\right) \\
\phi^\mp &\to \phi^\mp - \frac{1}{2}g\Lambda^\mp\left(H + 2\frac{M}{g} \pm i\phi^0\right) \mp \frac{i}{2}g\left(-\Lambda^3 + g_1\Lambda^0\right)\phi^\mp
\end{aligned}$$

Transformation to *physical* gauge parameters

$$\begin{aligned}
\Lambda^1 &= \frac{1}{\sqrt{2}}(\Lambda^+ + \Lambda^-), & \Lambda^2 &= \frac{i}{\sqrt{2}}(\Lambda^+ - \Lambda^-) \\
\Lambda^3 &= c_\theta\Lambda^Z + s_\theta\Lambda^A, & \Lambda^0 &= -s_\theta\Lambda^Z + c_\theta\Lambda^A \\
\Lambda^3 + g_1\Lambda^0 &= \frac{1}{c_\theta}\Lambda^Z, & -\Lambda^3 + g_1\Lambda^0 &= -\frac{c_\theta^2 - s_\theta^2}{c_\theta}\Lambda^Z - 2s_\theta\Lambda^A
\end{aligned}$$

In terms of physical parameters

$$\begin{aligned}
\phi^0 &\to \phi^0 - \frac{1}{2}g\frac{\Lambda^Z}{c_\theta}\left(H + 2\frac{M}{g}\right) + \frac{i}{2}g\left(\Lambda^-\phi^+ - \Lambda^+\phi^-\right) \\
\phi^\mp &\to \phi^- - \frac{1}{2}g\Lambda^\mp\left(H + 2\frac{M}{g} \pm i\phi^0\right) \pm \frac{i}{2}g\left(\frac{c_\theta^2 - s_\theta^2}{c_\theta}\Lambda^Z + 2s_\theta\Lambda^A\right)\phi^\mp \\
W^\mp_\mu &\to W^\mp_\mu \mp ig\Lambda^\mp\left(c_\theta Z_\mu + s_\theta A_\mu\right) \pm ig\left(c_\theta\Lambda^Z + s_\theta\Lambda^A\right)W^\mp_\mu - \partial_\mu\Lambda^\mp \\
A_\mu &\to A_\mu + igs_\theta\left(\Lambda^- W^+_\mu - \Lambda^+ W^-_\mu\right) - \partial_\mu\Lambda^A \\
Z_\mu &\to Z_\mu + igc_\theta\left(\Lambda^- W^+_\mu - \Lambda^+ W^-_\mu\right) - \partial_\mu\Lambda^Z \\
H &\to H + \frac{1}{2}\left(g\Lambda^3 + g_1\Lambda^0\right)\phi^0 + \frac{1}{2}\left(\Lambda^+ W^-_\mu + \Lambda^- W^+_\mu\right)
\end{aligned}$$

General form of the gauge transformations

$$\mathcal{C}^i \to \mathcal{C}^i + \left(\mathrm{M}^{ij} + g\mathrm{L}^{ij}\right)\Lambda^j, \qquad i = \pm, Z, A$$

Correspondence: of *physical* gauge parameters Λ^i
to *ghost fields* $X^i = X^+, X^-, Y^Z, Y^A$:

$$\begin{aligned} \Lambda^{\pm} &\to X^{\pm} \\ \Lambda^{Z} &\to Y^{Z} \\ \Lambda^{A} &\to Y^{A} \end{aligned}$$

In the charged sector we obtain:

$$\begin{aligned}
\mathcal{C}^- &= -\frac{1}{\xi}\partial_\mu W^-_\mu + \xi M \phi^- \\
&\to \mathcal{C}^- - \frac{1}{\xi}\partial_\mu\{-ig\Lambda^-\left(c_\theta Z_\mu + s_\theta A_\mu\right) + ig\left(c_\theta\Lambda^Z + s_\theta\Lambda^A\right)W^-_\mu - \partial_\mu\Lambda^-\} \\
&\quad +g\xi M\left\{-\frac{1}{2}\Lambda^-\left(H + 2\frac{M}{g} + i\phi^0\right) + \frac{i}{2}\frac{c_\theta^2 - s_\theta^2}{c_\theta}\Lambda^Z\phi^- + is_\theta\Lambda^A\phi^-\right\} \\
&= \mathcal{C}^- + \frac{1}{\xi}\Box\Lambda^- - \xi M^2\Lambda^- + \frac{i}{\xi}g\partial_\mu\{\Lambda^-\left(c_\theta Z_\mu + s_\theta A_\mu\right)\} \\
&\quad -\frac{i}{\xi}g\partial_\mu\{\left(c_\theta\Lambda^Z + s_\theta\Lambda^A\right)W^-_\mu\} - \frac{1}{2}\xi gM\left(H + i\phi^0\right)\Lambda^- \\
&\quad +\frac{i}{2}\xi gM\frac{c_\theta^2 - s_\theta^2}{c_\theta}\Lambda^Z\phi^- + i\xi g s_\theta M\Lambda^A\phi^-
\end{aligned}$$

and a similar one for $\mathcal{C}^+$.

The gauge invariance $\mathcal{C}^\pm = -\frac{1}{\xi}\partial_\mu W^-_\mu + \xi M\phi^- \to \mathcal{C}^\pm$ is restored if $\Lambda^\pm$ are identified with ghost fields $X^\pm$ with propagators

$$\frac{1}{\xi}\Box - \xi M^2$$

$X^\pm$

$$\frac{\xi}{p^2 + \xi^2 M^2}$$

and interactions

$$g\overline{X}^{\pm}\mathrm{L}^{\pm j}X^j, \qquad j = \pm, Z, A$$

Where we introduced four more fields: $\overline{X}^i = \overline{X}^+, \overline{X}^-, \overline{Y}^Z, \overline{Y}^A$.

For the transformation of $\mathcal{C}^A$ we obtain:

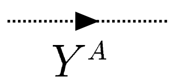
$$\begin{aligned}\mathcal{C}^A &= -\frac{1}{\xi_A}\partial_\mu A_\mu \to \mathcal{C}^A - \frac{1}{\xi_A}\partial_\mu[igs_\theta\left(\Lambda^- W_\mu^+ - \Lambda^+ W_\mu^-\right) - \partial_\mu \Lambda^A] \\ &= \mathcal{C}^A + \frac{1}{\xi_A}\Box\Lambda^A - \frac{i}{\xi_A} g s_\theta \partial_\mu \left(\Lambda^- W_\mu^+ - \Lambda^+ W_\mu^-\right)\end{aligned}$$

The gauge invariance is restored, if after identification

$$\frac{1}{\xi_A}\Box Y^A = \frac{i}{\xi_A} g s_\theta \partial_\mu \left(X^- W_\mu^+ - X^+ W_\mu^-\right)$$

i.e. Y^A has the propagator

$$\frac{1}{\xi_A}\Box$$



$$\frac{\xi_A}{p^2}$$

and interaction

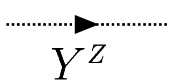
$$g\overline{Y}^A \mathrm{L}^{Aj} X^j, \qquad j = \pm, Z, A$$

For the transformation of $\mathcal{C}^Z$:

$$\begin{aligned}\mathcal{C}^Z &= -\frac{1}{\xi_Z}\partial_\mu Z_\mu + \xi_Z \frac{M}{c_\theta}\phi^0 \\ &\quad \to \mathcal{C}^Z - \frac{1}{\xi_Z}\partial_\mu\{igc_\theta\left(\Lambda^- W_\mu^+ - \Lambda^+ W_\mu^-\right) - \partial_\mu \Lambda^Z\} \\ &\quad +\xi_Z \frac{M}{c_\theta}\left\{-\frac{M}{c_\theta}\Lambda^Z - \frac{1}{2} g \frac{\Lambda^Z}{c_\theta} H + \frac{i}{2} g \left(\Lambda^- \phi^+ - \Lambda^+ \phi^-\right)\right\} \\ &= \mathcal{C}^Z \frac{1}{\xi_Z} + \Box\Lambda^Z - \xi_Z \frac{M^2}{c_\theta^2}\Lambda^Z - \frac{i}{\xi_Z} g c_\theta \partial_\mu \left(\Lambda^- W_\mu^+ - \Lambda^+ W_\mu^-\right) \\ &\quad -\frac{1}{2}\xi_Z g \frac{M}{c_\theta^2}\Lambda^Z H + i\xi_Z g \frac{M}{c_\theta}\left(\Lambda^- \phi^+ - \Lambda^+ \phi^-\right)\end{aligned}$$

giving the propagator of Y^Z

$$\frac{1}{\xi_Z}\Box - \xi_Z \frac{M^2}{c_\theta^2}$$



$$\frac{\xi_Z}{p^2 + \xi_Z^2 \frac{M^2}{c_\theta^2}}$$

and interaction

$$g\overline{Y}^Z \mathrm{L}^{Zj} X^j, \qquad j = \pm, Z, A$$

The complete interaction Lagrangian in the FP sector of SM derives trivially from above considerations:

$$
\begin{aligned}
\mathcal{L}^{\mathrm{I}}_{\mathrm{gf}} = \quad & igc_\theta W^+_\mu \left[\frac{1}{\xi_Z}\left(\partial_\mu \overline{Y}^Z\right) X^- - \frac{1}{\xi}\left(\partial_\mu \overline{X}^+\right) Y^Z\right] \\
& + igc_\theta W^-_\mu \left[\frac{1}{\xi}\left(\partial_\mu \overline{X}^-\right) Y^Z - \frac{1}{\xi_Z}\left(\partial_\mu \overline{Y}^Z\right) X^+\right] \\
& + igs_\theta W^+_\mu \left[\frac{1}{\xi_A}\left(\partial_\mu \overline{Y}^A\right) X^- - \frac{1}{\xi}\left(\partial_\mu \overline{X}^+\right) Y^A\right] \\
& + igs_\theta W^-_\mu \left[\frac{1}{\xi}\left(\partial_\mu \overline{X}^-\right) Y^A - \frac{1}{\xi_A}\left(\partial_\mu \overline{Y}^A\right) X^+\right] \\
& + igc_\theta \frac{1}{\xi} Z_\mu \left(\partial_\mu \overline{X}^+ X^+ - \partial_\mu \overline{X}^- X^-\right) \\
& + igs_\theta \frac{1}{\xi} A_\mu \left(\partial_\mu \overline{X}^+ X^+ - \partial_\mu \overline{X}^- X^-\right) \\
& - \frac{1}{2} gMH \left(\xi \overline{X}^+ X^+ + \xi \overline{X}^- X^- + \frac{\xi_Z}{c^2_\theta} \overline{Y}^Z Y^Z\right) \\
& - ig\xi M \frac{c^2_\theta - s^2_\theta}{c_\theta} \left(\overline{X}^+ Y^Z \phi^+ - \overline{X}^- Y^Z \phi^-\right) \\
& + \frac{i}{2} g\xi_Z M \frac{1}{c_\theta} \left(\overline{Y}^Z X^- \phi^+ - \overline{Y}^Z X^+ \phi^-\right) \\
& + igs_\theta \xi M \left(\overline{X}^- Y^A \phi^- - \overline{X}^+ Y^A \phi^+\right) \\
& + \frac{i}{2} g\xi M \left(\overline{X}^+ X^+ \phi^0 - \overline{X}^- X^- \phi^0\right)
\end{aligned}
$$

Note trivial rules:
– $\overline{Y}^Z$ and $\overline{Y}^A$ are accompanied by ξ_Z and ξ_A, correspondingly;
– $\overline{X}^\pm$ – by ξ^2;
– terms $\overline{Y}^Z X^-$ and $\overline{Y}^Z X^+$ or $\overline{X}^+ X^+$ and $\overline{X}^- X^-$ differ by sign for interactions with all fields but H.

Ghosts are fields satisfying Klein-Gordon equation.
They possess a charge resembling the fermionic charge.

Standard Model Lagrangian, Scalar Sector

The interactions in the scalar sector is given by the scalar potential

$$\mathcal{L}_{\mathrm{s}}^{\mathrm{I}} = -\mu^2 K^+ K - \frac{1}{2}\lambda \left(K^+ K\right)^2$$

where

$$K = \frac{1}{\sqrt{2}} \begin{pmatrix} H + \langle v \rangle + i\phi^0, \\ \\ i\sqrt{2}\phi^- \end{pmatrix}, \qquad \langle v \rangle = 2\frac{M}{g}$$

$$K^+ = \frac{1}{\sqrt{2}} \left(H + \langle v \rangle - i\phi^0 \, , -i\sqrt{2}\phi^+\right)$$

and for $\mathcal{L}_{\mathrm{s}}^{\mathrm{I}}$ we derive

$$\begin{aligned} \mathcal{L}_{\mathrm{s}}^{\mathrm{I}} = & -\frac{1}{2}\mu^2 \left[H^2 + 2\langle v \rangle H + \langle v \rangle^2 + \left(\phi^0\right)^2 + 2\phi^+\phi^-\right] \\ & -\frac{1}{8}\lambda \left\{H^4 + 4\langle v \rangle H^3 + 6\langle v \rangle^2 H^2 + 4\langle v \rangle^3 H + \langle v \rangle^4 + \left(\phi^0\right)^4 + 4\left(\phi^+\phi^-\right)^2 \right. \\ & \left. +2\left(H^2 + 2\langle v \rangle H + \langle v \rangle^2\right)\left[\left(\phi^0\right)^2 + 2\phi^+\phi^-\right] + 4\left(\phi^0\right)^2 \phi^+\phi^-\right\} \end{aligned}$$

Collect some selected terms:

constant term, $\quad -\frac{\langle v \rangle^2}{2}\left(\mu^2 + \frac{1}{4}\lambda\langle v \rangle^2\right)$, irrelevant

linear term, $H \quad -\langle v \rangle \left(\mu^2 + \frac{1}{2}\lambda\langle v \rangle^2\right) = -\langle v \rangle \beta_H$, vacuum tadpole

quadratic term, $H^2 \quad -\frac{1}{2}\left(\mu^2 + \frac{1}{2}\lambda\langle v \rangle^2 + \lambda\langle v \rangle^2\right) = -\frac{1}{2}\left(\beta_H + M_H^2\right)$

$\left[\left(\phi^0\right)^2 + 2\phi^+\phi^-\right] \quad -\frac{1}{2}\left(\mu^2 + \frac{1}{2}\lambda\langle v \rangle^2\right)$

where convenient set of parameters are

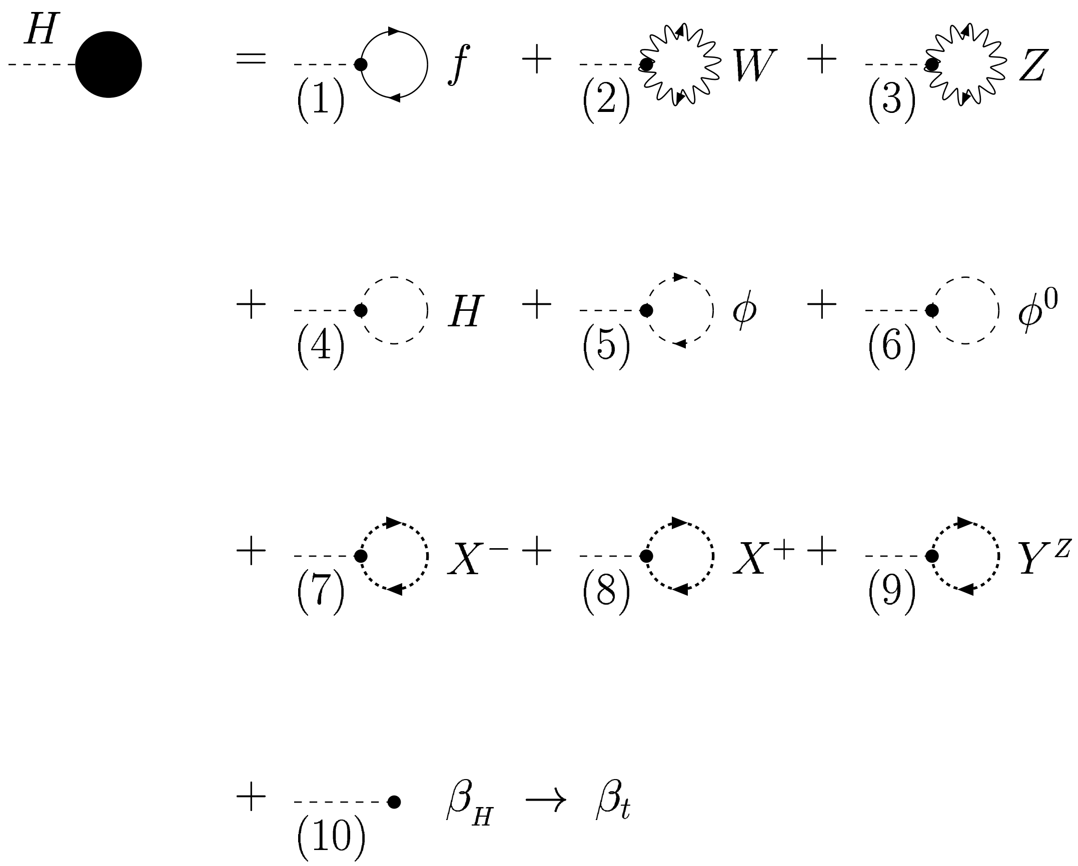
$$\beta_H = \mu^2 + 2\frac{\lambda}{g^2}M^2, \qquad \lambda = \frac{g^2 M_H^2}{4M^2} = g^2 \alpha_H, \qquad \alpha_H = \frac{1}{4}\frac{M_H^2}{M^2}$$

M_H – a measurable quantity; λ – from g, M, M_H; α_H – from M_H, M
μ^2 or β_H – should be treated as a new parameter, which has to be adjusted such that the vacuum expectation value of the H field remains zero order by order in perturbation theory.

The interaction Lagrangian (omitting irrelevant constant) and mass term $-\frac{1}{2}M_H^2 H^2$

$$
\begin{aligned}
\mathcal{L}_{\mathrm{S}}^{\mathrm{I}} \;=\; & -\beta_H \left\{ 2\frac{M}{g}H + \frac{1}{2}\left[H^2 + \left(\phi^0\right)^2 + 2\phi^+\phi^- \right] \right\} \\
& - g\alpha_H M \left[H^3 + H\left(\phi^0\right)^2 + 2H\phi^+\phi^- \right] \\
& - \frac{1}{8}g^2\alpha_H \Big[H^4 + \left(\phi^0\right)^4 + 2H^2\left(\phi^0\right)^2 \\
& + 4H^2\phi^+\phi^- + 4\left(\phi^0\right)^2\phi^+\phi^- + 4\left(\phi^+\phi^-\right)^2 \Big]
\end{aligned}
$$

Tadpoles and their role in proving gauge invariance of building blocks



More about tadpoles

In the Lagrangian, a tadpole constant should appear that is zero in the lowest order, and must be adjusted in such a way that the vacuum expectation value of the H field remains zero order by order in perturbation theory.

In order to exploit this option fully, we have to *renormalize* the vacuum expectation value itself:

$$K = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi \\ \sqrt{2} i \phi^- \end{pmatrix}, \qquad \chi = H + 2\frac{M}{g}\left(1 + g^2 \beta_t\right) + i\phi^0$$

Now we set $\mu^2 + 2\left(\lambda/g^2\right) M^2 = 0$ and, in turn, it is β_t that one fixes by the requirement of a zero vacuum expectation value of the H field. The $\mathcal{L}_{\rm s}^{\rm I}$ part of the Lagrangian becomes:

$$\begin{aligned}
\mathcal{L}_{\rm s}^{\rm I} \;=\; & -2\, g M M_{_H}^2 \beta_t\, H - \frac{1}{2} M_{_H}^2 \left(1 + 3\, g^2 \beta_t\right) H^2 \\
& -\frac{1}{2}\, g^2 M_{_H}^2 \beta_t \left[\left(\phi^0\right)^2 + 2\phi^+\phi^-\right] - g \alpha_{_H} M \left[H^3 + H\left(\phi^0\right)^2 + 2H\phi^+\phi^-\right] \\
& -\frac{1}{8} g^2 \alpha_{_H} \left[H^4 + \left(\phi^0\right)^4 + 2H^2\left(\phi^0\right)^2 \right. \\
& \left. + 4H^2\phi^+\phi^- + 4\left(\phi^0\right)^2 \phi^+\phi^- + 4\left(\phi^+\phi^-\right)^2\right]
\end{aligned}$$

Note that the only practical difference appears in the H^2 term.

From the renormalization of $\langle v \rangle$ we are automatically led to the addition of tadpoles in the $W - W$ and $Z - Z$ self-energies and in the corresponding vector–scalar transitions:

$$\begin{aligned}
& -g^2 \beta_t \left(M_{_0}^2\, Z_\mu Z_\mu + 2\, M^2 W_\mu^+ W_\mu^-\right) \\
& -g^2\, M \beta_t \left(\frac{1}{c_\theta}\, \phi^0 \partial_\mu Z_\mu + \phi^+ \partial_\mu W_\mu^- + \phi^- \partial_\mu W_\mu^+\right)
\end{aligned}$$

They are essential for proving that W, Z and H the self-energies are ξ-independent on their mass shells: $p^2 = -M^2$, $p^2 = -M_{_0}^2$, $p^2 = -M_{_H}^2$.

Interactions of fermions with gauge fields

A generic fermion-isodoublet

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix}, \qquad \psi_{L,R} = \frac{1}{2}(1 \pm \gamma_5)\,\psi$$

with a covariant derivative for the L-fields

$$D_\mu \psi_L \;=\; \left(\partial_\mu + g B^i_\mu T^i\right)\psi_L\,, \qquad i = 0, ..., 3$$

written in terms of generators of $SU(2) \otimes U(1)$:

$$T^a \;=\; -\frac{i}{2}\tau^a, \qquad T^0 \;=\; -\frac{i}{2} g_2 I$$

For the R-fields we have

$$D_\mu \psi_R \;=\; \left(\partial_\mu + g B^i_\mu t^i\right)\psi_R\,, \qquad i = 0, ..., 3$$

$$t^a \;=\; 0, \qquad t^0 = -\frac{i}{2}\begin{pmatrix} g_3 & 0 \\ 0 & g_4 \end{pmatrix}$$

Thus, ψ_L transforms as a doublet under $SU(2)$ and the ψ_R as a singlet. The parameters g_2, g_3 and g_4 are arbitrary hypercharges to be fixed below. The kinetic part of the Lagrangian can be written as

$$\mathcal{L}^{\mathrm{fer,I}}_V = -\overline{\psi}_L \not{D}\psi_L - \overline{\psi}_R \not{D}\psi_R\,, \qquad g_i = -\frac{s_\theta}{c_\theta}\lambda_i$$

<u>Exercise.</u> Consider only (3,0) components:

$$\begin{aligned}
&- \;\overline{\psi}_L\gamma_\mu\left(\partial_\mu - \frac{i}{2} g g_2 B^0_\mu I - \frac{i}{2} g B^3_\mu \tau^3\right)\psi_L - \overline{\psi}_R\gamma_\mu\left(\partial_\mu - \frac{i}{2} g B^0_\mu \begin{pmatrix} g_3 & 0 \\ 0 & g_4 \end{pmatrix}\right)\psi_R \\
&= \; -(\overline{u},\overline{d})_L\gamma_\mu\left[\partial_\mu - \frac{i}{2} g g_2 B^0_\mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{i}{2} g B^3_\mu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]\begin{pmatrix} u \\ d \end{pmatrix}_L \\
&\qquad -(\overline{u},\overline{d})_R\gamma_\mu\left[\partial_\mu - \frac{i}{2} g g_2 B^0_\mu \begin{pmatrix} g_3 & 0 \\ 0 & g_4 \end{pmatrix}\right]\begin{pmatrix} u \\ d \end{pmatrix}_R \\
&= \; -\overline{f}_L \not{\partial} f_L - \overline{f}_R \not{\partial} f_R + \frac{i}{2} g g_2 B^0_\mu \left(\overline{u}_L\gamma_\mu u_L + \overline{d}_L\gamma_\mu d_L\right) \\
&\qquad + \frac{i}{2} g B^3_\mu \left(\overline{u}_L\gamma_\mu u_L - \overline{d}_L\gamma_\mu d_L\right) + \frac{i}{2} g B^0_\mu \left(g_3\overline{u}_R\gamma_\mu u_R + g_4\overline{d}_R\gamma_\mu d_R\right) \rightarrow
\end{aligned}$$

$$\overline{f}_L\gamma_\mu f_L = \frac{1}{2}\overline{f}\gamma_\mu\left(1+\gamma_5\right)f, \qquad \overline{f}_R\gamma_\mu f_R = \frac{1}{2}\overline{f}\gamma_\mu\left(1-\gamma_5\right)f$$

$$\to \; -\overline{f}\partial\!\!\!/ f + \frac{i}{4}gg_2\left[\overline{u}\gamma_\mu\left(1+\gamma_5\right)u + \overline{d}\gamma_\mu\left(1+\gamma_5\right)d\right]\left(-s_\theta Z_\mu + c_\theta A_\mu\right)$$
$$+\frac{i}{4}g\left[\overline{u}\gamma_\mu\left(1+\gamma_5\right)u - \overline{d}\gamma_\mu\left(1+\gamma_5\right)d\right]\left(c_\theta Z_\mu + s_\theta A_\mu\right)$$
$$+\frac{i}{4}g\left[g_3\overline{u}\gamma_\mu\left(1-\gamma_5\right)u + g_4\overline{d}\gamma_\mu\left(1-\gamma_5\right)d\right]\left(-s_\theta Z_\mu + c_\theta A_\mu\right)$$

Collect terms with A_μ

$$\frac{i}{4}g\Big\{c_\theta g_2\left[\overline{u}\gamma_\mu\left(1+\gamma_5\right)u + \overline{d}\gamma_\mu\left(1+\gamma_5\right)d\right]$$
$$+s_\theta\left[\overline{u}\gamma_\mu\left(1+\gamma_5\right)u - \overline{d}\gamma_\mu\left(1+\gamma_5\right)d\right]$$
$$+c_\theta\left[g_3\overline{u}\gamma_\mu\left(1-\gamma_5\right)u + g_4\overline{d}\gamma_\mu\left(1-\gamma_5\right)d\right]\Big\} \to$$

First we require absence of axial currents

$$c_\theta g_2 + s_\theta - c_\theta g_3 = 0, \qquad c_\theta g_2 - s_\theta - c_\theta g_4 = 0$$

$$g_2 - g_1 - g_3 = 0, \qquad g_2 + g_1 - g_4 = 0$$

$$g_i \;=\; -\frac{s_\theta}{c_\theta}\lambda_i$$

$$-\lambda_2 + 1 + \lambda_3 = 0, \qquad -\lambda_2 - 1 + \lambda_4 = 0$$

$$\to \frac{i}{4}gs_\theta\Big\{-\lambda_2\left[\overline{u}\gamma_\mu u + \overline{d}\gamma_\mu d\right] + \overline{u}\gamma_\mu u - \overline{d}\gamma_\mu d$$
$$+\left(1-\lambda_2\right)\overline{u}\gamma_\mu u - \left(1+\lambda_2\right)\overline{d}\gamma_\mu d\Big\}$$
$$= \frac{i}{2}gs_\theta\Big\{\left(1-\lambda_2\right)\overline{u}\gamma_\mu u - \left(1+\lambda_2\right)\overline{d}\gamma_\mu d\Big\} = ieQ_u\overline{u}\gamma_\mu u + ieQ_d\overline{d}\gamma_\mu d$$

Thus, the parameters λ_i are fixed by the requirement that the e.m. current has the conventional structure, $iQ_f e\overline{f}\gamma_\mu f$. The solution is

$$\lambda_2 = 1 - 2Q_u \;=\; -1 - 2Q_d, \qquad \lambda_3 = -2Q_u, \qquad \lambda_4 = -2Q_d$$

with the charges

$$Q_f = 2I_f^{(3)}|Q_f|$$

$W^{\pm}$ always couples to a $V+A$ current and $\mathcal{L}_V^{\rm fer,I}$ reads

$$\mathcal{L}_V^{\rm fer,I} = \sum_f \left[igs_\theta Q_f A_\mu \overline{f}\gamma_\mu f + i\frac{g}{2c_\theta} Z_\mu \overline{f}\gamma_\mu \left(I_f^{(3)} - 2Q_f s_\theta^2 + I_f^{(3)}\gamma_5 \right) f \right]$$
$$+ \sum_d \left[i\frac{g}{2\sqrt{2}} W_\mu^+ \overline{u}\gamma_\mu \left(1+\gamma_5\right) d + i\frac{g}{2\sqrt{2}} W_\mu^- \overline{d}\gamma_\mu \left(1+\gamma_5\right) u \right]$$

where the first sum runs over all fermions, f, and the second over all doublets, d, of the SM.

Fermion – Vector Boson Interaction in presence of mixing

$$\mathcal{L}_V^{\rm fer,I} = \sum_f \left[igs_\theta Q_f A_\mu \overline{f}\gamma_\mu f + i\frac{g}{2c_\theta} Z_\mu \overline{f}\gamma_\mu \left(v_f + a_f\gamma_5 \right) f \right]$$
$$+ i\frac{g}{2\sqrt{2}} W_\mu^+ \overline{U}\gamma_\mu \left(1+\gamma_5\right) CD + i\frac{g}{2\sqrt{2}} W_\mu^- \overline{D}\gamma_\mu \left(1+\gamma_5\right) C^\dagger U$$

C – fermionic mixing matrix, CKM+leptonic (neutrino)

$$v_f = I_f^{(3)} - 2Q_f s_\theta^2\,, \qquad a_f = I_f^{(3)}$$

Correspondence between physical and bare parameters

$$M_W \leftrightarrow M\,, \qquad M_Z \leftrightarrow M_0 \equiv \frac{M}{c_\theta}\,, \qquad s_W \leftrightarrow s_\theta$$

Tree-level identities for coupling constants and vector boson masses

$$s_W^2 = \frac{e^2}{g^2} = 1 - \frac{M_W^2}{M_Z^2}\,, \qquad e^2 = 4\pi\alpha$$

$\alpha = 1/137.0359895...$ fine structure constant

Interactions of fermions with scalar fields

We need not only the field K but its conjugate K^c too in order to give mass to the up- and down- fermions.

$$\begin{aligned} K &= \frac{1}{\sqrt{2}}\begin{pmatrix} \chi \\ \\ \sqrt{2}i\phi^- \end{pmatrix}, \qquad \chi = H + \langle v\rangle + i\phi^0 \\ &= \frac{1}{\sqrt{2}}\left(H + \langle v\rangle + i\phi^a\tau^a\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ K^c &= i\tau^2 K^* = -\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}i\phi^+ \\ \\ \chi^* \end{pmatrix} \\ &= -\frac{1}{\sqrt{2}}\left(H + \langle v\rangle + i\phi^a\tau^a\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

The corresponding part of the Lagrangian:

$$\mathcal{L}_S^{\text{fer}} = -\alpha_f \overline{\psi}_L K u_R - \beta_f \overline{\psi}_L K^c d_R + h.c.$$

K gives masses to up fermions

K^c gives masses to down fermions

Gauge transformations:

$$\begin{aligned} K &\to \left(1 - \frac{i}{2}g\Lambda^a(x)\tau^a - \frac{i}{2}gg_1\Lambda^0(x)I\right)K, \qquad \text{with} \quad g_1 = -\frac{s_\theta}{c_\theta} \\ K^c &\to \left(1 - \frac{i}{2}g\Lambda^a(x)\tau^a + \frac{i}{2}gg_1\Lambda^0(x)I\right)K^c \\ \psi_L' &\to \left(1 - \frac{i}{2}g\Lambda^a(x)\tau^a - \frac{i}{2}gg_2\Lambda^0(x)I\right)\psi_L \\ \psi_R' &\to \left(1 - \frac{i}{2}g\begin{pmatrix} g_3 & 0 \\ 0 & g_4 \end{pmatrix}\Lambda^0(x)\right)\psi_R \end{aligned}$$

and we immediately see that $\mathcal{L}_S^{\text{fer}}$ is gauge invariant if $g_2 = g_1 + g_3$ and $g_2 = -g_1 + g_4$, the identities which were already established from the requirement that the e.m. current has the conventional structure!

Substutute K and K^c into $\mathcal{L}_S^{\rm fer}$

$$-\alpha_f\overline{\psi}_L\frac{1}{\sqrt{2}}\begin{pmatrix} H+\langle v\rangle+i\phi^0 \\ \\ \sqrt{2}i\phi^- \end{pmatrix}u_R+\beta_f\overline{\psi}_L\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}i\phi^+ \\ \\ H+\langle v\rangle-i\phi^0 \end{pmatrix}d_R+h.c.$$

$$\begin{aligned}
= & -\frac{\alpha_f}{\sqrt{2}}\left[\left(H+\frac{2M}{g}\right)\overline{u}_Lu_R+i\overline{u}_Lu_R\phi^0+i\sqrt{2}\overline{d}_Lu_R\phi^-\right] \\
& -\frac{\alpha_f}{\sqrt{2}}\left[\left(H+\frac{2M}{g}\right)\overline{u}_Ru_L-i\overline{u}_Ru_L\phi^0-i\sqrt{2}\overline{u}_Rd_L\phi^+\right] \\
& +\frac{\beta_f}{\sqrt{2}}\left[\left(H+\frac{2M}{g}\right)\overline{d}_Ld_R-i\overline{d}_Ld_R\phi^0+i\sqrt{2}\overline{u}_Ld_R\phi^+\right] \\
& +\frac{\beta_f}{\sqrt{2}}\left[\left(H+\frac{2M}{g}\right)\overline{u}_Ru_L+i\overline{d}_Rd_L\phi^0-i\sqrt{2}\overline{d}_Ru_L\phi^-\right]\rightarrow
\end{aligned}$$

$$\overline{u}_Rd_L=\overline{u}\frac{1}{2}\left(1+\gamma_5\right)d,\qquad \overline{u}_Ld_R=\overline{u}\frac{1}{2}\left(1-\gamma_5\right)d,\qquad \text{etc.}$$

$$\begin{aligned}
\rightarrow & -\frac{\alpha_f}{\sqrt{2}}\left[\left(H+\frac{2M}{g}\right)\overline{u}u-i\overline{u}\gamma_5u\phi^0+\frac{i}{\sqrt{2}}\overline{d}\left(1-\gamma_5\right)u\phi^--\frac{i}{\sqrt{2}}\overline{u}\left(1+\gamma_5\right)d\phi^+\right] \\
& +\frac{\beta_f}{\sqrt{2}}\left[\left(H+\frac{2M}{g}\right)\overline{d}d+i\overline{d}\gamma_5d\phi^0+\frac{i}{\sqrt{2}}\overline{u}\left(1-\gamma_5\right)d\phi^+-\frac{i}{\sqrt{2}}\overline{d}\left(1+\gamma_5\right)u\phi^-\right]
\end{aligned}$$

The solution for the Yukawa couplings

$$\alpha_f=\frac{1}{\sqrt{2}}g\frac{m_u}{M},\qquad \beta_f=-\frac{1}{\sqrt{2}}g\frac{m_d}{M}$$

This part of the Lagrangian

$$\mathcal{L}_S^{\rm fer}=-\sum_f m_f\overline{f}f+\mathcal{L}_S^{\rm fer,I}$$

with an interaction

$$\begin{aligned}
\mathcal{L}_S^{\rm fer,I} = & \sum_d\left\{i\frac{g}{2\sqrt{2}}\phi^+\left[\frac{m_u}{M}\overline{u}\left(1+\gamma_5\right)d-\frac{m_d}{M}\overline{u}\left(1-\gamma_5\right)d\right]\right. \\
& \left.+i\frac{g}{2\sqrt{2}}\phi^-\left[\frac{m_d}{M}\overline{d}\left(1+\gamma_5\right)u-\frac{m_u}{M}\overline{d}\left(1-\gamma_5\right)u\right]\right\} \\
& +\sum_f\left(-\frac{1}{2}gH\frac{m_f}{M}\overline{f}f+igI_f^{(3)}\phi^0\frac{m_f}{M}\overline{f}\gamma_5f\right)
\end{aligned}$$

Fermion mixing

Rewrite the expression for $\mathcal{L}_S^{\text{fer}}$

$$\mathcal{L}_S^{\text{fer}} = -\frac{g}{\sqrt{2}M}\overline{\psi}_L m_u K u_R + \frac{g}{\sqrt{2}M}\overline{\psi}_L m_d K^c d_R + h.c.$$

It could be generalized

$$\begin{aligned}\mathcal{L}_S^{\text{fer}} &= -\frac{g}{\sqrt{2}M}\left(\overline{\psi}_L\right)_\alpha (\mathcal{M}^U)_{\alpha,\beta} K \left(u_R\right)_\beta \\ &\quad + \frac{g}{\sqrt{2}M}\left(\overline{\psi}_L\right)_\alpha (\mathcal{M}^D)_{\alpha,\beta} K^c \left(d_R\right)_\beta + h.c.\end{aligned}$$

$$\left(u_R\right)_\beta = \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \\ u \\ c \\ t \end{pmatrix}_R = U_R\,, \qquad \left(d_R\right)_\beta = \begin{pmatrix} e \\ \mu \\ \tau \\ d \\ s \\ b \end{pmatrix}_R = D_R$$

$$\left(\overline{\psi}_L\right)_\alpha = \left(\overline{\nu}_e, \overline{\nu}_\mu, \overline{\nu}_\tau, \overline{u}, \overline{c}, \overline{t}; \overline{e}, \overline{\mu}, \overline{\tau}, \overline{d}, \overline{s}, \overline{b}\right)_L = \left(\overline{U}, \overline{D}\right)_L$$

$$\mathcal{M}^U = \begin{pmatrix} \mathcal{M}_l^U & O \\ O & \mathcal{M}_q^U \end{pmatrix}, \qquad \mathcal{M}^D = \begin{pmatrix} \mathcal{M}_l^D & O \\ O & \mathcal{M}_q^D \end{pmatrix}$$

where $\mathcal{M}_{l,q}^{U,D}$ are arbitrary complex matrices and O is zero-matrix, all 3×3 matrices.

Substituting scalar fields K and K^c we get the mass term

$$\mathcal{L}_S^{\text{fer},m} = -\frac{g}{\sqrt{2}M}\overline{U}'_L \mathcal{M}^U U'_R - \frac{g}{\sqrt{2}M}\overline{D}'_L \mathcal{M}^D D'_R + h.c.$$

In order to reduce it to usual form, one has to *diagonalize* the four mass matrices. This may be achieved with *bi-unitary* transformations

$$\mathcal{M}^U = \mathcal{U}_L^+ m_u \mathcal{U}_R\,, \qquad \mathcal{M}^D = \mathcal{D}_L^+ m_d \mathcal{D}_R$$

where $\mathcal{U}_L$, $\mathcal{U}_R$, $\mathcal{D}_L$, $\mathcal{D}_R$ are four different *unitary* 6×6 matrices

$$\mathcal{U}_L = \begin{pmatrix} (\mathcal{U}_L)_l & O \\ O & (\mathcal{U}_L)_q \end{pmatrix}, \qquad \text{etc.}$$

Fields with primes U'_L, U'_R, D'_L, D'_R are *weak eigenstates*. Introducing *mass eigenstates*:

$$U_L = \mathcal{U}_L U'_L\,, \qquad U_R = \mathcal{U}_R U'_R\,, \qquad D_L = \mathcal{D}_L D'_L\,, \qquad D_R = \mathcal{D}_R D'_R$$

we arrive at a usual mass term of the Lagrangian

$$\mathcal{L}_S^{\text{fer},m} \;=\; -\frac{g}{\sqrt{2}M}\overline{U} m_u U - \frac{g}{\sqrt{2}M}\overline{D} m_d D$$

where m_u and m_d are *diagonal* matrices in 6-dimensional U and D spaces.

The interaction part

$$\begin{aligned}
\mathcal{L}_S^{\text{fer,I}} \;=\;& i\frac{g}{2\sqrt{2}}\phi^+\left[\frac{m_u}{M}\overline{U}\left(1+\gamma_5\right)CD - \frac{m_d}{M}\overline{U}\left(1-\gamma_5\right)CD\right] \\
&+ i\frac{g}{2\sqrt{2}}\phi^-\left[\frac{m_d}{M}\overline{D}\left(1+\gamma_5\right)C^\dagger U - \frac{m_u}{M}\overline{D}\left(1-\gamma_5\right)C^\dagger U\right] \\
&+ \sum_f\left(-\frac{1}{2}gH\frac{m_f}{M}\overline{f}f + igI_f^{(3)}\phi^0\frac{m_f}{M}\overline{f}\gamma_5 f\right)
\end{aligned}$$

contains in charge boson sector the mixing matrix

$$C = \begin{pmatrix} (\mathcal{U}_L)_l\,(\mathcal{D}_L^+)_l & O \\ O & (\mathcal{U}_L)_l\,(\mathcal{D}_L^+)_l \end{pmatrix} = \begin{pmatrix} C_l & O \\ O & C_q \end{pmatrix}$$

that is not not diagonal because $\mathcal{U}_L$ and $\mathcal{D}_L$ are different matrices.

Some conclusions:

- mixing arises very natural;
- C_q – the usual CKM-matrix characterizing by 4 real parameters;
- C_l – its analog in lepton sector;
- complete lepton-quark analogy: extended SM (ESM) is a very natural extension of conventional SM with massless neutrino;
- Dirac mass terms (refer to Bilenky's and Carena's lectures, for a discussion whether does this contradict to present experimental data or whether do we really have experimental indications of physics beyond ESM)

The QCD Lagrangian

Pleliminaries:

In QCD one uses eight 3×3 hermitian matrices λ^a, a direct generalization of the 2×2 Pauli matrices, which satisfy

$$\mathrm{Tr}\lambda^a = 0, \qquad \mathrm{Tr}\lambda^a\lambda^b = 2\,\delta_{ab}\ ,$$
$$\left[T^a, T^b\right] = f^{abc}\,T^c \qquad \left\{T^a, T^b\right\} = -i\,d^{abc}\,T^c - \frac{1}{3}\,\delta_{ab}$$

with $T^a = -i\lambda^a/2$. The SU(3) structure constants f are antisymmetric in all three indices and satisfy the Jacobi identity while the d are symmetric in all indices.

The QCD Lagrangian contains three pieces:

— the colour gluon Lagrangian, $\mathcal{L}_c$;

— the colour fermion Lagrangian, $\mathcal{L}_c^{\rm fer}$;

— the colour Faddeev-Popov Lagrangian, $\mathcal{L}_c^{\rm FP}$.

The indices $a, b, \ldots$ take the values $1, \ldots, 8$ corresponding to the eight gluons. The indices $i, j, \ldots$ take the values $1, \ldots, 3$, corresponding to three colours. An index σ designates the quark flavors: u, d, c, s, t, b.

$$\begin{aligned}
\mathcal{L}_c &= -\frac{1}{2}\,\partial_\nu G^a_\mu \partial_\nu G^a_\mu - g_{_S}\,f^{abc}\,\partial_\mu G^a_\nu G^b_\mu G^c_\nu \\
&\quad -\frac{1}{4}\,g_{_S}^2\,f^{abc} f^{ade}\,G^b_\mu G^c_\nu G^d_\mu G^e_\nu \\
\mathcal{L}_c^{\rm fer} &= \frac{1}{2} i g_{_S}\left(\overline{q}^\sigma_i \gamma^\mu \lambda^a_{ij} q^\sigma_j\right)\,G^a_\mu, \\
\mathcal{L}_c^{\rm FP} &= \overline{\kappa}^a \partial^2 \kappa^a + g_{_S}\,f^{abc}\,\partial_\mu \overline{\kappa}^a \kappa^b G^c_\mu
\end{aligned}$$

$g_{_S}$ – strong coupling constant and

$$\alpha_{_S} = \frac{g_{_S}^2}{4\,\pi}\ , \qquad a = \frac{\alpha_{_S}}{\pi}$$

$\overline{MS}$ scheme and the running parameters

$\overline{MS}$ scheme, unit mass μ scale,
$\rightarrow$ the renormalization group (RG) equations for an observable O

$$\mu^2 \frac{d}{d\mu^2} O\left(g_S, m, \mu\right) = 0$$

$\overline{MS}$ renormalized parameters depend on the scale μ, i.e. they run

$$\mu^2 \frac{d}{d\mu^2}\left[\frac{\alpha_S(\mu)}{\pi}\right]\Big|_{g^2_{S_0},\, m_0} = \beta(\alpha_S(\mu)) = -\sum_{N=0}^{\infty} \beta_N^{(n_f)} \left[\frac{\alpha_S(\mu)}{\pi}\right]^{N+2}$$

$$\begin{aligned}
\beta_0^{(n_f)} &= \frac{1}{4}\left(11 - \frac{2}{3}n_f\right), \qquad \beta_1^{(n_f)} = \frac{1}{16}\left(102 - \frac{38}{3}n_f\right), \\
\beta_2^{(n_f)} &= \frac{1}{64}\left(\frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2\right), \\
\beta_3^{(n_f)} &= \frac{1}{256}\left\{\frac{149753}{6} + 3564\zeta(3) - \left[\frac{1078361}{162} + \frac{6508}{27}\zeta(3)\right] n_f \right. \\
&\quad \left. + \left[\frac{50065}{162} + \frac{6472}{81}\zeta(3)\right] n_f^2 + \frac{1093}{729} n_f^3\right\}
\end{aligned}$$

where n_f is the number of active quarks (quarks with mass less than the energy scale μ).

The $\overline{MS}$ scheme is practically implemented within phenomenological applications of QCD by setting μ to a characteristic scale of a process.

Following the RG equation for α_S one could give its value at a fixed reference scale μ_0 but it is an established convention to introduce a dimensional parameter Λ.

The solution of the RG equation

$$\begin{aligned}
\frac{\alpha_S\left(\mu^2, \Lambda_{\overline{\rm MS}}\right)}{\pi} &= \frac{1}{\beta_0 L} - \frac{1}{(\beta_0 L)^2} b_1 \ln L + \frac{1}{(\beta_0 L)^3}\left[b_1^2\left(\ln^2 L - \ln L - 1\right) + b_2\right] \\
&\quad - \frac{1}{(\beta_0 L)^4}\left[b_1^3\left(L^3 - \frac{5}{2}L^2 - 2L + \frac{1}{2}\right) + 3 b_1 b_2 L - \frac{1}{2} b_3\right]
\end{aligned}$$

with coefficients,

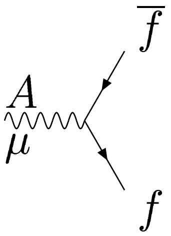
$$b_i = \frac{\beta_i}{\beta_0}, \qquad \text{and} \qquad L = \ln\frac{\mu^2}{\Lambda^2_{\overline{\rm MS}}}$$

Feynman Rules for Vertices

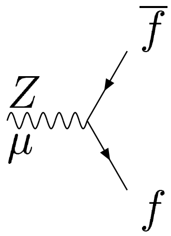
The arrow convention:

1. The arrows occurring in lines are denoting fermion lines, or the flow of electric charge or the flow of the FP-ghost quantum number. An incoming W^+ will therefore be denoted by an incoming arrow.
2. An arrow pointing inwards implies positive charge flowing into the vertex. For a negatively charged FP field the flow of charge is opposite to the direction of the arrow, for a positively charged FP field it is in the direction of the arrow.
3. In vertices all momenta are taken to be ingoing.

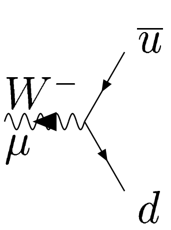
— Fermionic vertices



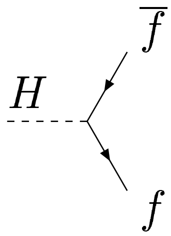
$$ieQ_f\gamma_\mu$$



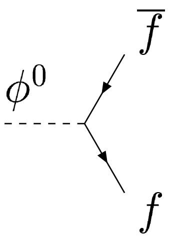
$$i\frac{g}{2c_\theta}\gamma_\mu\left(v_f + a_f\gamma_5\right)$$



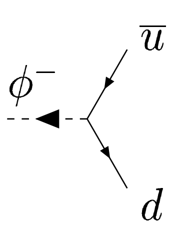
$$i\frac{g}{2\sqrt{2}}\gamma_\mu\left(1+\gamma_5\right)$$



$$-\frac{1}{2}\,g\frac{m_f}{M}$$

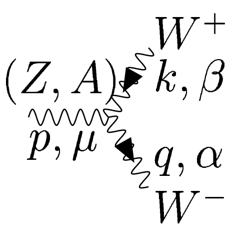


$$igI_f^{(3)}\frac{m_f}{M}\gamma_5$$

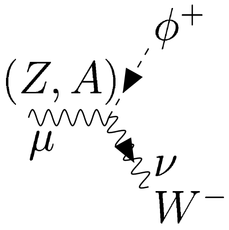


$$i\frac{g}{2\sqrt{2}}\left[\frac{m_d}{M}\left(1+\gamma_5\right) - \frac{m_u}{M}\left(1-\gamma_5\right)\right]$$

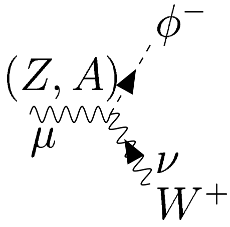
— Tri-linear vertices



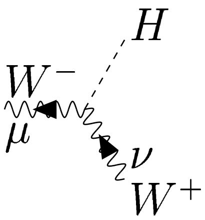
$$g\left(c_\theta, s_\theta\right)\left\{\delta_{\mu\alpha}\left(p-q\right)_\beta + \delta_{\alpha\beta}\left(q-k\right)_\mu + \delta_{\mu\beta}\left(k-p\right)_\alpha\right\}$$



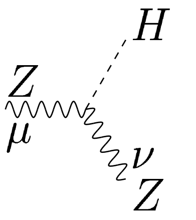
$$ig\left(\frac{s_\theta^2}{c_\theta}, -s_\theta\right) M\delta_{\mu\nu}$$



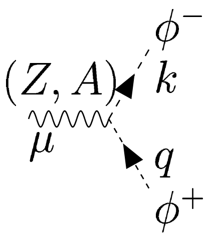
$$ig\left(-\frac{s_\theta^2}{c_\theta}, s_\theta\right) M\delta_{\mu\nu}$$



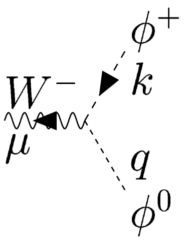
$$-gM\delta_{\mu\nu}$$



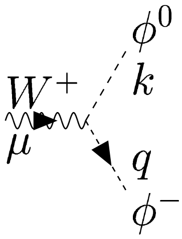
$$-g\frac{M}{c_\theta^2}\delta_{\mu\nu}$$



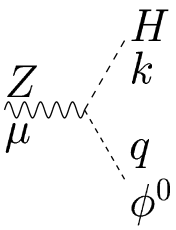
$$g\left(\frac{c_\theta^2 - s_\theta^2}{2c_\theta}, s_\theta\right)(q-k)_\mu$$



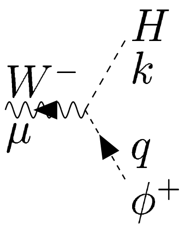
$$\frac{1}{2}g\,(q-k)_\mu$$



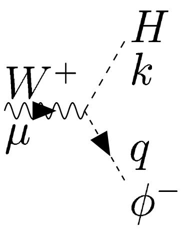
$$\frac{1}{2}g\,(q-k)_\mu$$



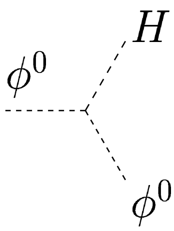
$$\frac{i}{2}\frac{g}{c_\theta}(q-k)_\mu$$



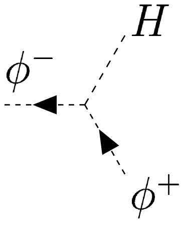
$$\frac{i}{2}g\,(q-k)_\mu$$



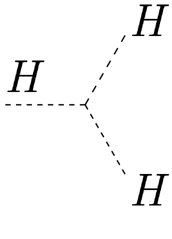
$$\frac{i}{2}g\,(q-k)_\mu$$



$$-\frac{1}{2}g\frac{M_H^2}{M}$$

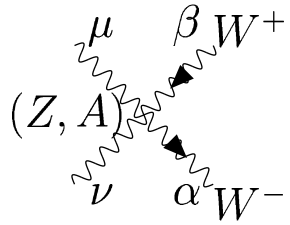


$$-\frac{1}{2}g\frac{M_H^2}{M}$$

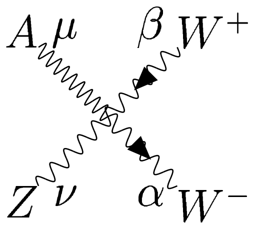


$$-\frac{3}{2}g\frac{M_H^2}{M}$$

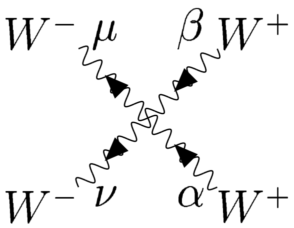
— Quadri-linear vertices



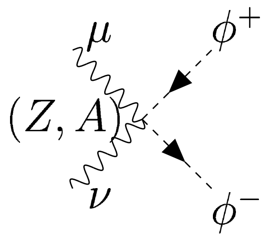
$$-g^2\left(c_\theta^2, s_\theta^2\right)\left\{2\delta_{\mu\nu}\delta_{\alpha\beta}-\delta_{\mu\alpha}\delta_{\nu\beta}-\delta_{\mu\beta}\delta_{\nu\alpha}\right\}$$



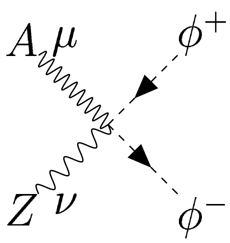
$$-g^2 s_\theta c_\theta\left\{2\delta_{\mu\nu}\delta_{\alpha\beta}-\delta_{\mu\alpha}\delta_{\nu\beta}-\delta_{\mu\beta}\delta_{\nu\alpha}\right\}$$



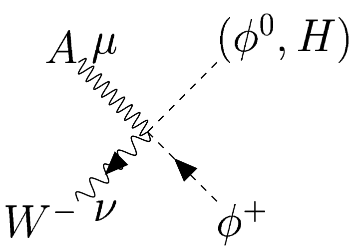
$$g^2\left\{2\delta_{\mu\nu}\delta_{\alpha\beta}-\delta_{\mu\alpha}\delta_{\nu\beta}-\delta_{\mu\beta}\delta_{\nu\alpha}\right\}$$



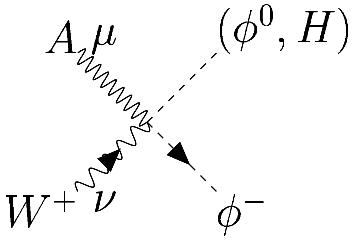
$$g^2\left(-\frac{\left(c_\theta^2-s_\theta^2\right)^2}{2c_\theta^2}, -2s_\theta^2\right)\delta_{\mu\nu}$$



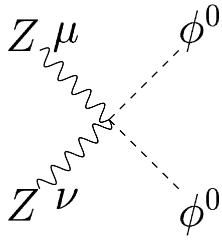
$$-g^2\frac{s_\theta}{c_\theta}\left(c_\theta^2-s_\theta^2\right)\delta_{\mu\nu}$$



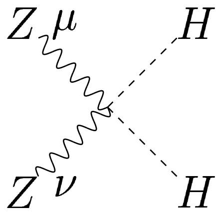
$$\frac{1}{2}\left(1,-i\right)g^2 s_\theta\delta_{\mu\nu}$$



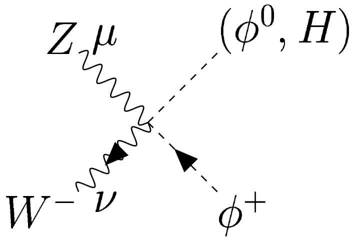
$$\frac{1}{2}\left(1,i\right)g^2 s_\theta \delta_{\mu\nu}$$



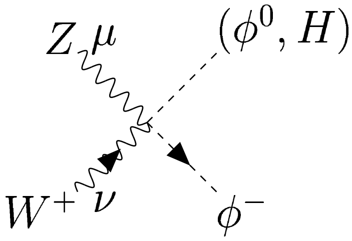
$$-\frac{1}{2}\frac{g^2}{c_\theta^2}\delta_{\mu\nu}$$



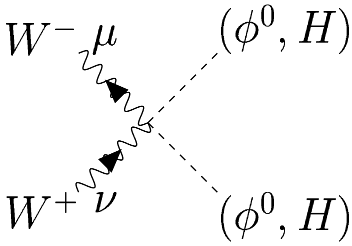
$$-\frac{1}{2}\frac{g^2}{c_\theta^2}\delta_{\mu\nu}$$



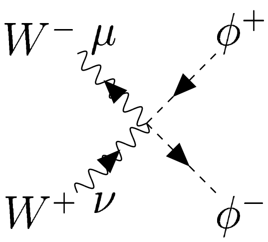
$$-\frac{1}{2}\left(1,-i\right)g^2\frac{s_\theta^2}{c_\theta}\delta_{\mu\nu}$$



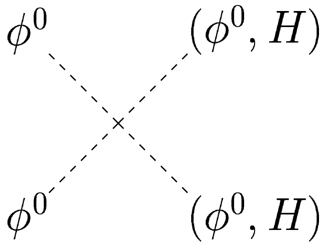
$$-\frac{1}{2}\left(1,i\right)g^2\frac{s_\theta^2}{c_\theta}\delta_{\mu\nu}$$



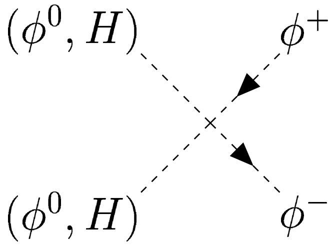
$$-\frac{1}{2}\left(1,1\right)g^2\delta_{\mu\nu}$$



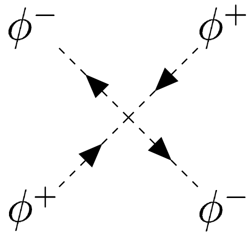
$$-\frac{1}{2}g^2\delta_{\mu\nu}$$



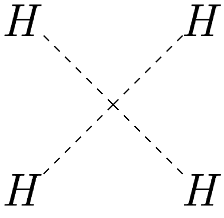
$$-\frac{1}{4}\left(3,1\right)g^2\frac{M_H^2}{M^2}$$



$$-\frac{1}{4}\left(1,1\right)g^2\frac{M_H^2}{M^2}$$

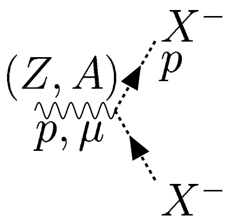


$$-\frac{1}{2}g^2\frac{M_H^2}{M^2}$$

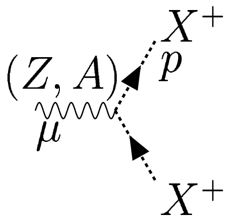


$$-\frac{3}{4}g^2\frac{M_H^2}{M^2}$$

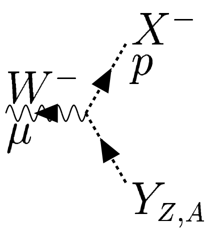
— Tri-linear vertices involving FP ghosts



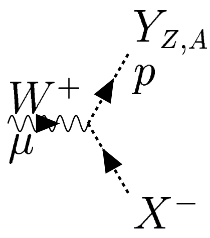
$$g\frac{1}{\xi}\left(c_\theta, s_\theta\right) p_\mu$$



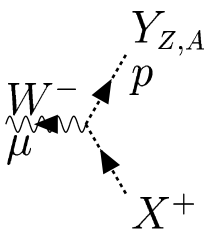
$$-g\frac{1}{\xi}\left(c_\theta, s_\theta\right) p_\mu$$



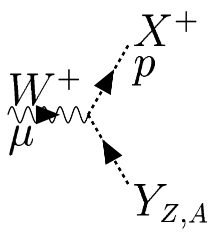
$$-g\frac{1}{\xi}\left(c_\theta, s_\theta\right) p_\mu$$



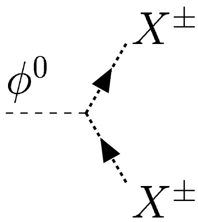
$$-g\left(\frac{c_\theta}{\xi_Z}, \frac{s_\theta}{\xi_A}\right) p_\mu$$



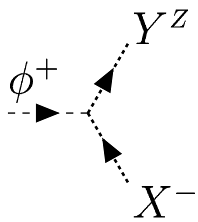
$$g\left(\frac{c_\theta}{\xi_Z}, \frac{s_\theta}{\xi_A}\right) p_\mu$$



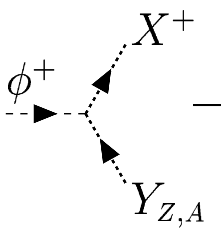
$$g\frac{1}{\xi}\left(c_\theta, s_\theta\right) p_\mu$$



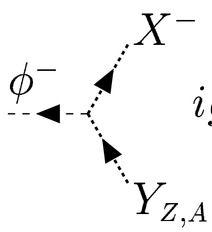
$$\pm\frac{i}{2}g\xi M$$



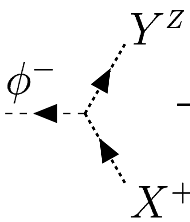
$$\frac{i}{2}\frac{g}{c_\theta}\xi_Z M$$



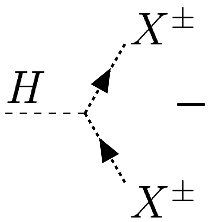
$$-ig\left(\frac{c_\theta^2 - s_\theta^2}{2c_\theta}, s_\theta\right)\xi M$$



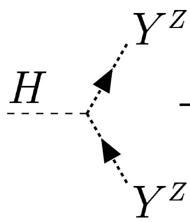
$$ig\left(\frac{c_\theta^2 - s_\theta^2}{2c_\theta}, s_\theta\right)\xi M$$



$$-\frac{i}{2}\frac{g}{c_\theta}\xi_Z M$$



$$-\frac{1}{2}g\xi M$$



$$-\frac{1}{2}\frac{g}{c_\theta^2}\xi_Z M$$

Summary of Level 1

Standard Model, its Fields and Lagrangian

Gauges:
General R_ξ, with three parameters, ξ, ξ_Z ,ξ_A
t'Hooft-Feynman or Renormalizable, all $\xi = 1$
Physical or Unitary, $\xi \to \infty, \xi_Z \to \infty, \xi_A = 1$
Gauge Invariance, ξ independence

Feynman Rules

Ready to build Diagrams:

– tree level diagrams
– one(many)-loop diagrams

Pauli Metrics
on-mass-shell momentum

$$p^2 \;=\; -M^2 \quad \to \quad \text{propagator} \quad \sim \frac{1}{p^2 + M^2}$$

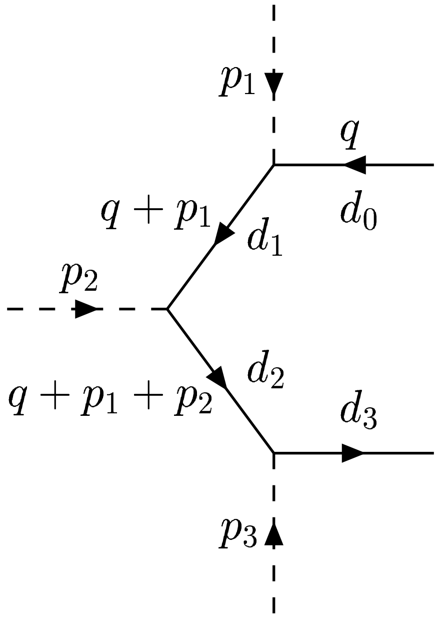
left projector

$$\gamma_L = \frac{1 + \gamma_5}{2}$$

Feynman parametrization

$$\begin{aligned}
\frac{1}{AB} &= \int_0^1 dx \frac{1}{[Ax+B(1-x)]^2} \\
\frac{1}{A^2B} &= \int_0^1 dx \frac{2x}{[Ax+B(1-x)]^3} \\
\frac{1}{ABC} &= \int_0^1 dyy \int_0^1 dx \frac{1}{\{A(1-y)+y[Bx+C(1-x)]\}^3} \\
\frac{1}{ABCD} &= \cdots
\end{aligned}$$

N-point function (arrows indicate momentum flow)



with

$$d_i = (q + p_1 + \ldots + p_i)^2 + m_{i+1}^2 - i\epsilon$$

is reducable to

$$\ldots \int_0^1 dyy \int_0^1 dx \frac{1, q_\mu, q_\mu q_\nu, \ldots}{\left(q^2 - 2q \cdot p_{x,y,\ldots} + m_{x,y,\ldots}^2 - i\epsilon\right)^\alpha}$$

where $\alpha = N$ for N-point function.
The quantities $p_{x,y,\ldots}$ and $m_{x,y,\ldots}^2$ are linear combinations of external momenta, p_i, and internal masses, m_i^2, p_i^2 and $(p_j + \ldots + p_{j+k})^2$, correspondingly.

Basics of Dimension Regularization

All is based on only one integral

$$\mu^{4-n}\int d^n q \,\frac{1}{\left(q^2+m^2-i\epsilon\right)^{\alpha}} = i\pi^{\frac{n}{2}}\frac{\Gamma\left(\alpha-\frac{n}{2}\right)}{\Gamma\left(\alpha\right)}\left(\frac{m^2}{\mu^2}\right)^{\frac{n}{2}-\alpha}$$

by shift $q \to q-p$, one derives

$$J(p) = \mu^{4-n}\int d^n q \,\frac{1}{\left(q^2-2q\cdot p+m^2-i\epsilon\right)^{\alpha}} = i\pi^{\frac{n}{2}}\frac{\Gamma\left(\alpha-\frac{n}{2}\right)}{\Gamma\left(\alpha\right)}\left(\frac{m^2-p^2}{\mu^2}\right)^{\frac{n}{2}-\alpha}$$

by differentiating $\partial_\mu J(p) \;\to$ $\quad [x\Gamma\left(x\right)=\Gamma\left(x+1\right)$ and $\alpha+1\to\alpha]$

$$\mu^{4-n}\int d^n q \,\frac{q_\mu}{\left(q^2-2q\cdot p+m^2-i\epsilon\right)^{\alpha}} = i\pi^{\frac{n}{2}}\frac{\Gamma\left(\alpha-\frac{n}{2}\right)}{\Gamma\left(\alpha\right)}\left(\frac{m^2-p^2}{\mu^2}\right)^{\frac{n}{2}-\alpha} p_\mu$$

by one more differentiation $\;\to$

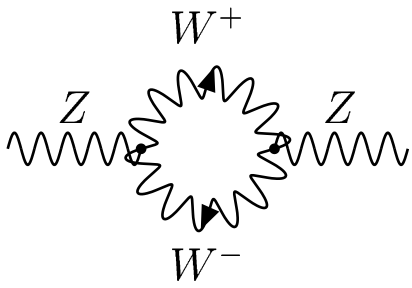
$$\mu^{4-n}\int d^n q \,\frac{q_\mu q_\nu}{\left(q^2-2q\cdot p+m^2-i\epsilon\right)^{\alpha}} = i\pi^{\frac{n}{2}}\frac{1}{\Gamma\left(\alpha\right)}\left(\frac{m^2-p^2}{\mu^2}\right)^{\frac{n}{2}-\alpha} \times\left[\frac{1}{2}\delta_{\mu\nu}\left(m^2-p^2\right)\Gamma\left(\alpha-1-\frac{n}{2}\right)+p_\mu p_\nu\Gamma\left(\alpha-\frac{n}{2}\right)\right]$$

Particular case

$$\mu^{4-n}\int d^n q \,\frac{1}{\left(q^2\right)^{\alpha}} = 0$$

Divergences counting: poles versus powers

Ultraviolet divergences



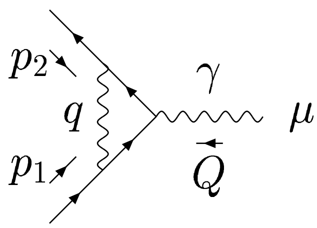
$$\int d^n q \, \frac{\left(\delta_{\mu\nu} + \frac{q_\mu q_\nu}{M^2}\right)\left[\delta_{\alpha\beta} + \frac{(q+p)_\alpha (q+p)_\beta}{M^2}\right]}{(q^2+M^2)\left[(q+p)^2+M^2\right]}$$

$$\int d^n q \, \frac{\left(\delta_{\mu\nu}\delta_{\alpha\beta} + \frac{q_\mu q_\nu}{M^2}\delta_{\alpha\beta} + \frac{q_\mu q_\nu q_\alpha q_\beta}{M^4} + \dots\right)}{(q^2+M^2)\left[(q+p)^2+M^2\right]}$$

$$\ln\Lambda \qquad \Lambda^2 \qquad \Lambda^4$$

$$\frac{1}{n-4} \qquad \frac{1}{n-2} \qquad \frac{1}{n-0}$$

Infrared divergences



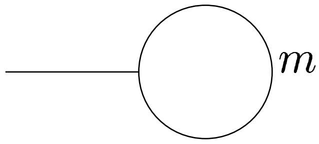
With $p_1^2 = p_2^2 = -m^2$ and $Q^2 = (p_1+p_2)^2$

$$\Lambda_\mu \sim \int d^n q \frac{-4p_1\cdot p_2\gamma_\mu + 2\left(\not p_1\gamma_\alpha\gamma_\mu - \gamma_\mu\gamma_\alpha\not p_2\right)q_\alpha + (2-n)\,\gamma_\alpha\gamma_\mu\gamma_\beta q_\alpha q_\beta}{q^2\left[(q+p_1)^2+m^2\right]\left[(q-p_2)^2+m^2\right]}$$

$$= \int d^n q \frac{-4p_1\cdot p_2\gamma_\mu + 2\left(\not p_1\gamma_\alpha\gamma_\mu - \gamma_\mu\gamma_\alpha\not p_2\right)q_\alpha + (2-n)\,\gamma_\alpha\gamma_\mu\gamma_\beta q_\alpha q_\beta}{q^2[q^2+2q\cdot p_1][q^2-2q\cdot p_2]}$$

Scalar	Vector	Tensor
Infrared	Finite	Ultraviolet

One-point integrals, A-functions



Scalar one-point integral. Needed for tadpole diagrams and in the reduction of higher order functions

$$i\pi^2 A_0(m) = \mu^{4-n} \int d^n q \frac{1}{q^2 + m^2 - i\epsilon}$$

Using general formula with $\alpha = 1$

$$A_0(m) = \pi^{n/2-2} \Gamma\left(1 - \frac{n}{2}\right) m^2 \left(\frac{m^2}{\mu^2}\right)^{n/2-2}$$

If one introduces $\varepsilon = 4 - n$ and expands around $n = 4$

$$\Gamma\left(\frac{\varepsilon}{2}\right) = \frac{2}{\varepsilon} - \gamma, \qquad a^{\frac{\varepsilon}{2}} = 1 + \frac{\varepsilon}{2} \ln a$$

$$A_0(m) = m^2 \left(-\frac{2}{\varepsilon} + \gamma + \ln \pi - 1 + \ln \frac{m^2}{\mu^2}\right) + \mathcal{O}(\varepsilon)$$

$$\frac{1}{\bar{\varepsilon}} = \frac{2}{\varepsilon} - \gamma - \ln \pi$$

then

$$A_0(m) = m^2 \left(-\frac{1}{\bar{\varepsilon}} - 1 + \ln \frac{m^2}{\mu^2}\right) + \mathcal{O}(\varepsilon)$$

Tensor one-point integrals

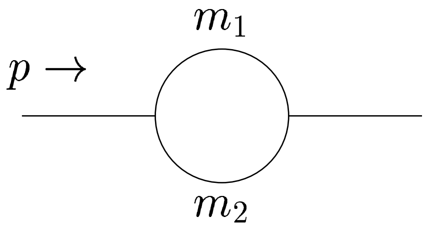
$$i\pi^2 A_{\mu\nu}(m) = \mu^{4-n} \int d^n q \frac{q_\mu q_\nu}{q^2 + m^2 - i\epsilon}$$

$$A_{\mu\nu}(m) = A_2(m)\, \delta_{\mu\nu}$$

$$A_2(m) = -\frac{1}{4} m^2 A_0(m) + \frac{1}{8} m^4$$

Rank four tensor integral may be reduced in a similar way

Two-point integrals, B-functions



Scalar two-point integral. Is met in the calculation of self-energy diagrams containing two propagators d_0 and d_1

$$i\pi^2 B_0\left(p^2; m_1, m_2\right) = \mu^{4-n} \int d^n q \frac{1}{d_0 d_1}$$

$$d_0 = q^2 + m_1^2 - i\epsilon, \qquad d_1 = (q+p)^2 + m_2^2 - i\epsilon$$

We will use the general expression for propagators

$$d_i = (q + p_1 + \ldots + p_i)^2 + m_{i+1}^2 - i\epsilon$$

For arbitrary internal masses the B_0 function is

$$B_0\left(p^2; m_1, m_2\right) = \frac{1}{\bar{\varepsilon}} - R - \ln\frac{m_1 m_2}{\mu^2} + \frac{m_1^2 - m_2^2}{2p^2} \ln\frac{m_1^2}{m_2^2} + 2$$

where $\Lambda^2 = \lambda\left(-p^2, m_1^2, m_2^2\right)$, $\qquad \lambda\left(x, y, z\right) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$ and

$$R = -\frac{\Lambda}{p^2} \ln\frac{p^2 - i\epsilon + m_1^2 + m_2^2 - \Lambda}{2\, m_1 m_2}$$

Particular cases:

1) if $m_1 = m_2 = m$

$$B_0\left(p^2; m, m\right) = \frac{1}{\bar{\varepsilon}} + 2 - \ln\frac{m^2}{\mu^2} - \beta \ln\frac{\beta+1}{\beta-1}, \qquad \beta^2 = 1 + \frac{4m^2}{p^2 - i\epsilon}$$

2) if one of the internal masses is zero

$$B_0\left(p^2; 0, m\right) = \frac{1}{\bar{\varepsilon}} + 2 - \ln\frac{m^2}{\mu^2} - \left(1 + \frac{m^2}{p^2}\right) \ln\left(1 + \frac{p^2 - i\epsilon}{m^2}\right)$$

3) if both internal lines are massless

$$B_0\left(p^2; 0, 0\right) = \frac{1}{\bar{\varepsilon}} + 2 - \ln\frac{p^2 - i\epsilon}{\mu^2}$$

B_0 function develops imaginary part above the physical threshold, $s = -p^2 \geq (m_1 + m_2)^2$

$$\mathrm{Im} B_0\left(p^2; m_1, m_2\right) \;=\; i\pi \frac{\sqrt{\lambda\left(s, m_1^2, m_2^2\right)}}{s}\,\theta\left(s - (m_1+m_2)^2\right)$$

<u>Tensor two-point integrals.</u>

Reduction to linear combinations of scalar functions

$$i\pi^2 B_\mu\left(p^2; m_1, m_2\right) = \mu^{4-n} \int d^n q \frac{q_\mu}{d_0 d_1} = i\pi^2 B_1\left(p^2; m_1, m_2\right)\, p_\mu$$

Using the relation $q^2 = d_0 - m_1^2$

$$q \cdot p = \frac{1}{2}\left(d_1 - d_0 + f_1^b\right), \qquad f_1^b = -p^2 + m_1^2 - m_2^2$$

we derive the identity

$$p^2\, B_1\left(p^2; m_1, m_2\right) \;=\; \frac{1}{2}\left[A_0\left(m_1\right) - A_0\left(m_2\right) + f_1^b B_0\left(p^2; m_1, m_2\right)\right]$$

The function B_1 obeys the symmetry

$$B_1\left(p^2; m_2, m_1\right) = -B_1\left(p^2; m_1, m_2\right) - B_0\left(p^2; m_1, m_2\right)$$

The rank two tensor integral can be reduced as follows:

$$\begin{aligned} i\pi^2 B_{\mu\nu}\left(p^2; m_1, m_2\right) \;&=\; \mu^{4-n} \int d^n q \frac{q_\mu q_\nu}{d_0 d_1} \\ &=\; i\pi^2\left[B_{21}\left(p^2; m_1, m_2\right) p_\mu p_\nu + B_{22}\left(p^2; m_1, m_2\right)\delta_{\mu\nu}\right] \end{aligned}$$

The last relation can be multiplied by $\delta_{\mu\nu}$ and by p_ν to give

$$\begin{aligned} p^2 B_{21}\left(p^2; m_1, m_2\right) + n B_{22}\left(p^2; m_1, m_2\right) \;&=\; A_0\left(m_2\right) - m_1^2 B_0\left(p^2; m_1, m_2\right) \\ p^2 B_{21}\left(p^2; m_1, m_2\right) + B_{22}\left(p^2; m_1, m_2\right) \;&=\; \frac{1}{2}\left[A_0\left(m_2\right) + f_1^b B_1\left(p^2; m_1, m_2\right)\right] \end{aligned}$$

In order to solve this system of equations we have to compute the singular parts of the scalar one and two point functions in terms of the quantity $\frac{1}{\bar{\varepsilon}}$. We will need a function χ

$$\chi(x) = -p^2 x^2 + \left(p^2 + m_2^2 - m_1^2\right) x + m_1^2 - i\epsilon$$

A simple calculation shows that

$$
\begin{aligned}
B_0\left(p^2;m_1,m_2\right) &= \frac{1}{\bar{\varepsilon}}-\int_0^1 dx\ln\left(\frac{\chi}{\mu^2}\right)\overset{\text{sing}}{\to}\frac{1}{\bar{\varepsilon}}\\
B_1\left(p^2;m_1,m_2\right) &= -\frac{1}{2}\frac{1}{\bar{\varepsilon}}+\int_0^1 dxx\ln\left(\frac{\chi}{\mu^2}\right)\overset{\text{sing}}{\to}-\frac{1}{2}\frac{1}{\bar{\varepsilon}}\\
B_{21}\left(p^2;m_1,m_2\right) &= \frac{1}{3}\frac{1}{\bar{\varepsilon}}-\int_0^1 dxx^2\ln\left(\frac{\chi}{\mu^2}\right)\overset{\text{sing}}{\to}\frac{1}{3}\frac{1}{\bar{\varepsilon}}\\
B_{22}\left(p^2;m_1,m_2\right) &= -\frac{1}{2}\left(\frac{1}{\bar{\varepsilon}}+1\right)\int_0^1 dx\chi+\frac{1}{2}\int_0^1 dx\chi\ln\left(\frac{\chi}{\mu^2}\right)\\
&\overset{\text{sing}}{\to} -\frac{1}{4}\left(m_1^2+m_2^2+\frac{1}{3}p^2\right)\frac{1}{\bar{\varepsilon}}
\end{aligned}
$$

By using these relations we arrive at a system of equations (*) with

$$
\begin{aligned}
n\,B_{22}\left(p^2;m_1,m_2\right) &= 4\,B_{22}\left(p^2;m_1,m_2\right)+\frac{K^2}{6}\\
K^2 &= p^2+3\left(m_1^2+m_2^2\right)
\end{aligned}
$$

Introduce the matrix

$$
X_2=\begin{pmatrix} p^2 & 4\\ p^2 & 1\end{pmatrix}
$$

and the vector b with components

$$
\begin{aligned}
b_1 &= A_0\left(m_2\right)-m_1^2B_0\left(p^2;m_1,m_2\right)-\frac{K^2}{6}\\
b_2 &= \frac{1}{2}\left[A_0\left(m_2\right)+f_1^bB_1\left(p^2;m_1,m_2\right)\right]
\end{aligned}
$$

The $B_{2i}\left(p^2;m_1,m_2\right)$ functions can therefore be obtained by inversion

$$
B_{2i}\left(p^2;m_1,m_2\right)=\left[X_2\right]_{ij}^{-1}\,b_j
$$

List of the final results

$$
\begin{aligned}
B_1\left(p^2;m_1,m_2\right) &= \frac{1}{2p^2}\left[A_0\left(m_1\right)-A_0\left(m_2\right)+\left(\Delta m^2-p^2\right)B_0\left(p^2;m_1,m_2\right)\right]\\
B_{21}\left(p^2;m_1,m_2\right) &= \frac{3\left(m_1^2+m_2^2\right)+p^2}{18p^2}\\
&\quad+\frac{\Delta m^2-p^2}{3p^4}A_0\left(m_1\right)-\frac{\Delta m^2-2p^2}{3p^4}A_0\left(m_2\right)\\
&\quad+\frac{\lambda\left(-p^2,m_1^2,m_2^2\right)-3p^2m_1^2}{3p^4}B_0\left(p^2;m_1,m_2\right)\\
B_{22}\left(p^2;m_1,m_2\right) &= -\frac{3\left(m_1^2+m_2^2\right)+p^2}{18}\\
&\quad-\frac{\Delta m^2-p^2}{12\,p^2}A_0\left(m_1\right)+\frac{\Delta m^2+p^2}{12\,p^2}A_0\left(m_2\right)\\
&\quad-\frac{\lambda\left(-p^2,m_1^2,m_2^2\right)}{12p^2}B_0\left(p^2;m_1,m_2\right)\\
\Delta m^2 &= m_1^2-m_2^2
\end{aligned}
$$

Reduction for $p^2=0$

The reduction algorithm fails at $p^2 = 0$. In this case, the explicit expressions should be derived from the defining integral representation.

$$
\begin{aligned}
B_0\left(0;m_1,m_2\right) &= \frac{A_0\left(m_2\right)-A_0\left(m_1\right)}{m_1^2-m_2^2}\\
B_1\left(0;m_1,m_2\right) &= -\frac{1}{2}B_0\left(0;m_1,m_2\right)+\frac{1}{2}\left(m_1^2-m_2^2\right)B_{0p}\left(0;m_1,m_2\right)\\
B_{0p}\left(0;m_1,m_2\right) &= \left.\frac{\partial B_0\left(p^2;m_1,m_2\right)}{\partial p^2}\right|_{p^2=0}\\
B_{22}\left(0;m_1,m_2\right) &= -\frac{1}{4}\left(m_1^2+m_2^2\right)\left(\frac{1}{\bar{\varepsilon}}-\ln\frac{m_1m_2}{\mu^2}+\frac{3}{2}\right)+\frac{m_1^4+m_2^4}{8\left(m_1^2-m_2^2\right)}\\
&\quad\times\ln\frac{m_1^2}{m_2^2}
\end{aligned}
$$

Derivatives of B-functions

In actual calcuation one needs also derivatives of B-functions. They appear in renormalization factors associated with external lines

$$\begin{aligned}\frac{\partial B_{\{0;1;21\}}}{\partial p^2} &= -\int_0^1 dx \frac{\{x;-x^2;x^3\}(1-x)}{\chi} \\ \frac{\partial B_{22}}{\partial p^2} &= -\frac{1}{12}\frac{1}{\bar{\varepsilon}} + \frac{1}{2}\int_0^1 dx x(1-x)\ln\left(\frac{\chi}{\mu^2}\right)\end{aligned}$$

For the QED corrections, the derivatives are infrared divergent and must be regulated

$$\begin{aligned}B_0\left(p^2;m,0\right) &= \pi^{n/2-2}\,\Gamma\left(2-\frac{n}{2}\right)\int_0^1 dx\left(\frac{\chi}{\mu^2}\right)^{n/2-2} \\ \chi(x) &= (1-x)\left(p^2x+m^2\right)\end{aligned}$$

With $n = 4+\varepsilon'$ we derive

$$\begin{aligned}\frac{\partial}{\partial p^2}B_0\left(p^2;m,0\right) &= -\pi^{\varepsilon'/2}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\int_0^1 dx\frac{x(1-x)}{\chi(x)}\left(\frac{\chi(x)}{\mu^2}\right)^{\varepsilon'/2} \\ \frac{\partial}{\partial p^2}B_0\left(p^2;m,0\right)\Big|_{p^2=-m^2} &= -\pi^{\varepsilon'/2}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\frac{1}{m^2}\left(\frac{m^2}{\mu^2}\right)^{\varepsilon'/2}\left(\frac{1}{\varepsilon'}-\frac{1}{1+\varepsilon'}\right)\end{aligned}$$

Expanding the various terms in ε' we derive

$$\frac{\partial}{\partial p^2}B_0\left(p^2;m,0\right)\Big|_{p^2=-m^2} = -\frac{1}{2m^2}\left(\frac{1}{\hat{\varepsilon}}-2+\ln\frac{m^2}{\mu^2}\right)$$

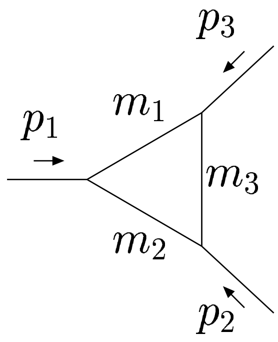
where

$$\frac{1}{\hat{\varepsilon}} = \frac{2}{\varepsilon'}+\gamma+\ln\pi = \frac{2}{n-4}+\gamma+\ln\pi = -\frac{1}{\bar{\varepsilon}}$$

Similarly one obtains the derivative of B_1

$$\begin{aligned}\frac{\partial}{\partial p^2}B_1\left(p^2;m,0\right)\Big|_{p^2=-m^2} &= \pi^{\varepsilon'/2}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\left(m^2\right)^{-1+\varepsilon'/2}\left(\mu^2\right)^{-\varepsilon'/2} \\ &\quad\times\left(\frac{1}{\varepsilon'}-\frac{2}{1+\varepsilon'}+\frac{1}{2+\varepsilon'}\right) \\ \frac{\partial}{\partial p^2}B_1\left(p^2;m,0\right)\Big|_{p^2=-m^2} &= \frac{1}{2m^2}\left(\frac{1}{\hat{\varepsilon}}-3+\ln\frac{m^2}{\mu^2}\right)\end{aligned}$$

Three-point integrals, C-functions



The scalar three-point functions are associated with vertex corrections, they are much more involved than the previous ones.

Basic definition and symmetry properties

$$i\pi^2\, C_0\left(p_1^2, p_2^2, Q^2; m_1, m_2, m_3\right) = \mu^{4-n} \int d^n q \frac{1}{d_0 d_1 d_2}$$

d_i in this case are

$$\begin{aligned} d_0 &= q^2 + m_1^2 - i\epsilon, \\ d_1 &= (q+p_1)^2 + m_2^2 - i\epsilon \\ d_2 &= (q+Q)^2 + m_3^2 - i\epsilon \end{aligned}$$

where $Q = p_1 + p_2$ and $Q^2 = (p_1 + p_2)^2$ is one of the Mandelstam variables, $Q^2 = -s, t$ or u, for an arbitrary $2 \to 2$ amplitude. In terms of a particular choice of Feynman parameters C_0 becomes

$$\begin{aligned} & C_0\left(p_1^2, p_2^2, Q^2; m_1, m_2, m_3\right) = \\ & = \int_0^1 dx \int_0^x dy \left(ax^2 + by^2 + cxy + dx + ey + f\right)^{-1} \end{aligned}$$

$$\begin{aligned} a &= -p_2^2, \qquad b = -p_1^2 \\ c &= p_1^2 + p_2^2 - Q^2, \qquad d = p_2^2 + m_2^2 - m_3^2 \\ e &= -p_2^2 + Q^2 + m_1^2 - m_2^2, \qquad f = m_3^2 - i\epsilon \end{aligned}$$

The scalar C_0 function is invariant under simultaneous cyclic permutations in the two sets of arguments: $\{p_1^2 p_2^2 Q^2\}$ and $\{m_1 m_2 m_3\}$.

Some particular cases of C_0-functions

There is one generic three-point scalar integral which occurs in the calculation of two fermion production when all fermionic masses, but top-quark mass, are neglected. It corresponds to the following choice:

$$p_{1,2}^2 = 0, \quad (p_1+p_2)^2 = Q^2, \quad m_1 = M_1\ , \quad m_2 = M_2\ , \quad m_3 = M_3\ .$$

Then the coefficients in the quadratic form become

$$\begin{aligned} a &= 0, & b &= 0, & c &= -Q^2 \\ d &= M_2^2 - M_3^2, & e &= Q^2 + M_1^2 - M_2^2, & f &= M_3^2 - i\epsilon \end{aligned}$$

and the result for C_0

$$C_0\left(0,0,Q^2;M_1,M_2,M_3\right) = \int_0^1 dx \int_0^x dy \frac{1}{\chi(x,y)}$$

where the function χ is a quadratic form in the x and y,

$$\chi(x,y) = Q^2 y\,(1-x) + M_1^2 y + M_2^2\,(x-y) + M_3^2\,(1-x)$$

In this particular case we get

$$C_0 = \frac{1}{Q^2}\sum_{i=1}^{3}(-1)^{\delta_{i3}}\left[\mathrm{Li}_2\left(\frac{x_0-1}{x_0-x_i}\right) - \mathrm{Li}_2\left(\frac{x_0}{x_0-x_i}\right)\right]$$

with four different roots

$$\begin{aligned} x_0 &= 1 + \frac{M_1^2 - M_2^2}{Q^2}\ , \qquad x_3 = \frac{M_3^2}{M_3^2 - M_2^2} \\ x_{1,2} &= \frac{Q^2 + M_1^2 - M_3^2 \mp \sqrt{\lambda\left(-Q^2, M_1^2, M_3^2\right)}}{2Q^2} \end{aligned}$$

And dilogarithm function

$$\mathrm{Li}_2\,(x) = \int_0^1 dy \frac{\ln\,(1-xy)}{y}$$

All masses squared are understood with equal infinitesimal imaginary parts: $M_i^2 \to M_i^2 - i\epsilon$, necessary to properly define the analytic continuation at $Q^2 \to -s$.

The special cases which are met in realistic calculation:

$$
\begin{aligned}
C_0\left(0,0,Q^2;M_1,0,M_3\right) &= \frac{1}{Q^2}\ln\frac{x_2}{x_2-1}\ln\frac{x_1-1}{x_1} \\
C_0\left(0,0,Q^2;M_1,0,M_1\right) &= \frac{1}{Q^2}\ln^2\frac{\beta_Q+1}{\beta_Q-1}\,, \qquad \beta_Q=\sqrt{1+\frac{4M_1^2}{Q^2}} \\
C_0\left(0,0,Q^2;M_1,M_2,0\right) &= C_0\left(0,0,Q^2;0,M_2,M_1\right) \\
&= \frac{1}{Q^2}\left[\mathrm{Li}_2\left(1-\frac{M_1^2}{M_2^2}\right)-\mathrm{Li}_2\left(1-\frac{Q^2+M_1^2}{M_2^2}\right)\right] \\
C_0\left(0,0,Q^2;0,M_2,0\right) &= \frac{1}{Q^2}\left[\mathrm{Li}_2\left(1\right)-\mathrm{Li}_2\left(1-\frac{Q^2}{M_2^2}\right)\right]
\end{aligned}
$$

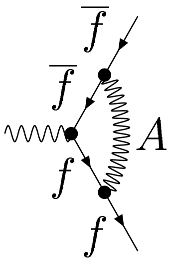
One more interesting case

$$
\begin{aligned}
C_0\left(-m^2,-m^2,Q^2;0,m,0\right) &= \frac{1}{m^2\left(y_1-y_2\right)}\left[2\mathrm{Li}_2\left(\frac{1}{y_1}\right)-2\mathrm{Li}_2\left(\frac{1}{y_2}\right)\right. \\
&\quad \left.+\mathrm{Li}_2\left(y_1\right)-\mathrm{Li}_2\left(y_2\right)\right], \\
y_{1,2} &= -\frac{m^2}{2Q^2}\left(1\pm\sqrt{1+\frac{4m^2}{Q^2}}\right)
\end{aligned}
$$

The scalar integral with all internal masses set to zero

$$
\begin{aligned}
C_0\left(p_1^2,p_2^2,Q^2;0,0,0\right) &= \frac{1}{Q^2\left(a_+-a_-\right)}\left[\ln\left(a_+a_-\right)\ln\frac{1-a_+}{1-a_-}\right. \\
&\quad \left.+2\mathrm{Li}_2\left(a_+\right)-2\mathrm{Li}_2\left(a_-\right)\right], \\
a_\pm &= \frac{Q^2+p_1^2-p_2^2\pm\sqrt{\lambda\left(Q^2,p_1^2,p_2^2\right)}}{2\,Q^2}
\end{aligned}
$$

Infrared divergent C_0 function



$p_1^2 = -m^2$, $p_2^2 = -m^2$ and $m_2 = \lambda$ with λ small with respect to all other quantities. Although by now the infrared singularities are treated within the dimensional regularization approach, this example is a useful bridge with the mass-regularization method.

$$C_0\left(-m^2, -m^2, Q^2; m, \lambda, m\right) = \int_0^1 dy \int_0^1 dx \frac{x}{\chi(x,y)}$$

with the integrand

$$\begin{aligned}\chi(x,y) &= x^2\chi(y) + \lambda^2(1-x) - i\epsilon \\ \chi(y) &= m^2(1-y) + m^2 y + Q^2 y(1-y)\end{aligned}$$

Using

$$\int_0^1 dx \frac{x}{\chi(y)x^2 + \lambda^2(1-x)} = \frac{1}{2\chi(y)} \ln\left(\frac{\chi(y)}{\lambda^2}\right) + \mathcal{O}\left(\frac{\lambda}{\sqrt{\chi(y)}}\right)$$

one obtains the following decomposition

$$\begin{aligned}C_0 &= F_1 \ln\left(\frac{\mu}{\lambda}\right) + \frac{1}{2}F_2 \\ F_1 &= \int_0^1 dy\, \frac{1}{\chi(y)} = \frac{2}{Q^2\beta_m} \ln\frac{\beta_m+1}{\beta_m-1} \\ F_2 &= \int_0^1 dy\, \frac{1}{\chi(y)} \ln\frac{\chi(y)}{\mu^2} = F_1\, \ln\left(\frac{Q^2 - i\varepsilon}{\mu^2}\right) \\ &+ \frac{1}{Q^2\beta_m}\left[\ln\frac{\beta_m+1}{\beta_m-1}\ln\frac{m^2\beta_m^2}{Q^2} - 2\mathrm{Li}_2\left(\frac{\beta_m+1}{2\beta_m}\right) + 2\mathrm{Li}_2\left(\frac{\beta_m-1}{2\beta_m}\right)\right]\end{aligned}$$

The bridge to dimensional regularization

$$\ln\left(\frac{\mu}{\lambda}\right)^2 \leftrightarrow \frac{1}{\hat{\varepsilon}}$$

General case of the C_0-function

The result is obtained from the defining equation (**) as follows: first we do the shift of integration variable $y \to y + \alpha x$, where α is one of the two roots of

$$-p_1^2\alpha^2 - 2p_1 \cdot p_2\alpha - p_2^2 = 0$$

e.g.

$$\alpha = \frac{p_1 \cdot p_2 + \sqrt{\Delta_3}}{-p_1^2}$$

where $\Delta_3 = (p_1 \cdot p_2)^2 - p_1^2 p_2^2$ is the corresponding Gram determinant. After this trick, the denominator becomes linear in x and with one trick more, namely reordering of integration,

$$\int_0^1 dx \int_{-\alpha x}^{(1-\alpha)x} dy = \int_0^{1-\alpha} dx \int_{y/(1-\alpha)}^1 dx - \int_0^{-\alpha} dy \int_{-y/\alpha}^1 dx$$

it may be readily integrated over x. In evaluating this scalar function we define the auxiliary quantities:

$$\begin{aligned} N &= 2\sqrt{\Delta_3}\,, \qquad r_0 = \frac{-d - e\alpha}{N} \\ y_0 &= -\frac{r_0}{\alpha}\,, \qquad y_1 = \frac{r_0}{1-\alpha}\,, \qquad y_2 = r_0 + \alpha \end{aligned}$$

and introduce three different *pinches* of the basic three-point integral:

$$\begin{aligned} P^{(0)}(y) &= -p_2^2 y^2 + \left(p_2^2 + m_2^2 - m_3^2\right) y + m_3^2 - i\epsilon \\ P^{(1)}(y) &= -Q^2 y^2 + \left(Q^2 + m_1^2 - m_3^2\right) y + m_3^2 - i\epsilon \\ P^{(2)}(y) &= -p_1^2 y^2 + \left(p_1^2 + m_1^2 - m_2^2\right) y + m_2^2 - i\epsilon \end{aligned}$$

After some tediuos but strightforward algebra, we obtain in terms of the pinches the one fold integral representation

$$C_0\left(p_1^2, p_2^2, Q^2; m_1, m_2, m_3\right) = \frac{1}{N} \sum_{i=0}^{2} \int_0^1 dy \, \frac{1}{y - y_i} \left[\ln P^{(i)}(y) - \ln P^{(i)}(y_i)\right]$$

Eventually $\to$ 12 dilogarithms + Veltman's η-functions

Reduction of vector three-point integral

The rank one tensor

$$i\pi^2 C_\mu\left(p_1^2, p_2^2, Q^2; m_1, m_2, m_3\right) = \mu^{4-n} \int d^n q \, \frac{q_\mu}{d_0 d_1 d_2}$$

and its decomposition:

$$i\pi^2 \left[C_{11}\left(p_1^2, p_2^2, Q^2; m_1, m_2, m_3\right) p_{1\mu} + C_{12}\left(p_1^2, p_2^2, Q^2; m_1, m_2, m_3\right) p_{2\mu}\right]$$

The reduction is based on the relations

$$\begin{aligned} p_1 \cdot q &= \frac{1}{2}\left(d_1 - d_0 + f_1^c\right), \qquad p_2 \cdot q = \frac{1}{2}\left(d_2 - d_1 + f_2^c\right) \\ f_1^c &= -p_1^2 + m_1^2 - m_2^2 \,, \qquad f_2^c = -Q^2 + p_1^2 + m_2^2 - m_3^2 \end{aligned}$$

Pinches are needed:

$$\begin{aligned} C_k^{(0)} &= B_k(1,2) = B_k\left(p_2^2; m_2, m_3\right) \\ C_k^{(1)} &= B_k(0,2) = B_k\left(Q^2; m_1, m_3\right) \\ C_k^{(2)} &= B_k(0,1) = B_k\left(p_1^2; m_1, m_2\right) \end{aligned}$$

where k runs over all possible indices of the B_k functions, i.e. $0, 1, 21, 22$. For instance

$$i\pi^2 B_0(i,j) = \mu^{4-n} \int d^n q \frac{1}{d_i' d_j'}$$

$$d_i' = d_i \,, \qquad d_j' = d_j \,, \qquad \text{for} \quad i = 0$$

$$d_1' = q^2 + m_2^2 \,, \qquad d_2' = (q + p_2)^2 + m_3^2$$

As we did for the two-point integrals a matrix is introduced,

$$X_{3,ij} = p_i \cdot p_j$$

which satisfies $\det X_3 = -\Delta_3$, and also the vector $R_{12}^{(1)}$

$$R_{12}^{(1)} = \frac{1}{2} \begin{pmatrix} C_0^{(1)} - C_0^{(0)} + f_1^c C_0 \\ C_0^{(2)} - C_0^{(1)} + f_2^c C_0 \end{pmatrix}$$

With their help we finally derive

$$C_{1i}\left(p_1^2, p_2^2, Q^2; m_1, m_2, m_3\right) = \left(X_3^{-1}\right)_{ij} \, R_{12\,j}^{(1)}$$

Second rank tensor reduction

The rank two tensor integral

$$i\pi^2 C_{\mu\nu} = \mu^{4-n} \int d^n q \frac{q_\mu q_\nu}{d_0 d_1 d_2}$$

and its decomposition

$$i\pi^2 \left[C_{21} p_{1\mu} p_{1\nu} + C_{22} p_{2\mu} p_{2\nu} + C_{23} \{p_1 p_2\}_{\mu\nu} + C_{24} \delta_{\mu\nu}\right]$$

where the symmetrized combination is introduced

$$\{p_1 p_2\}_{\mu\nu} = p_{1\mu} p_{2\nu} + p_{1\nu} p_{2\mu}$$

Multiply both equations above by $\delta_{\mu\nu}$, by $p_{1\nu}$ or by $p_{2\nu}$

$$\begin{aligned}
\mu^{4-n} \int d^n q \frac{q^2}{d_0 d_1 d_2} &= i\pi^2 \Big(C_{21} p_1^2 + C_{22} p_2^2 + 2 C_{23} p_1 \cdot p_2 + n C_{24}\Big) \qquad (*) \\
\mu^{4-n} \int d^n q \frac{q \cdot p_1 q_\mu}{d_0 d_1 d_2} &= i\pi^2 \Big[\Big(C_{21} p_1^2 + C_{23} p_1 \cdot p_2 + C_{24}\Big) p_{1\mu} \\
&\quad + \Big(C_{22} p_1 \cdot p_2 + C_{23} p_1^2\Big) p_{2\mu}\Big] \\
\mu^{4-n} \int d^n q \frac{q \cdot p_2 q_\mu}{d_0 d_1 d_2} &= i\pi^2 \Big[\Big(C_{21} p_1 \cdot p_2 + C_{23} p_2^2\Big) p_{1\mu} \\
&\quad + \Big(C_{22} p_2^2 + C_{23} p_1 \cdot p_2 + C_{24}\Big) p_{2\mu}\Big]
\end{aligned}$$

Then

$$\mu^{4-n} \int d^n q \frac{q \cdot p_1 q_\mu}{d_0 d_1 d_2} = \frac{1}{2} \mu^{4-n} \int d^n q \left(\frac{q_\mu}{d_0 d_2} - \frac{q_\mu - p_{1\mu}}{d'_1 d'_2} + f_1^c \frac{q_\mu}{d_0 d_1 d_2}\right)$$
$$= \frac{i}{2}\pi^2 \left[C_1^{(1)} Q_\mu - C_1^{(0)} p_{2\mu} + C_0^{(0)} p_{1\mu} + f_1^c \left(C_{11} p_{1\mu} + C_{12} p_{2\mu}\right)\right]$$

and

$$\mu^{4-n} \int d^n q \frac{q \cdot p_2 q_\mu}{d_0 d_1 d_2} = \frac{1}{2} \mu^{4-n} \int d^n q \left(\frac{q_\mu}{d_0 d_1} - \frac{q_\mu}{d_0 d_2} + f_2^c \frac{q_\mu}{d_0 d_1 d_2}\right)$$
$$= \frac{i}{2}\pi^2 \left[C_1^{(2)} p_{1\mu} - C_1^{(1)} Q_\mu + f_2^c \left(C_{11} p_{1\mu} + C_{12} p_{2\mu}\right)\right]$$

The solution to the reduction consists in defining two vectors

$$R_{13}^{(2)} = \frac{1}{2}\begin{pmatrix} C_1^{(1)} + C_0^{(0)} + f_1^c C_{11} - 2C_{24} \\ C_1^{(2)} - C_1^{(1)} + f_2^c C_{11} \end{pmatrix}$$

and

$$R_{32}^{(2)} = \frac{1}{2}\begin{pmatrix} C_1^{(1)} - C_1^{(0)} + f_1^c C_{12} \\ -C_1^{(1)} + f_2^c C_{12} - 2C_{24} \end{pmatrix}$$

and again a simple inversion:

$$\begin{pmatrix} C_{21} \\ C_{23} \end{pmatrix} = X_3^{-1}\, R_{13}^{(2)}, \qquad \text{and} \qquad \begin{pmatrix} C_{23} \\ C_{22} \end{pmatrix} = X_3^{-1}\, R_{32}^{(2)}.$$

Note, that in the derivation there are more equations than unknowns, this provides an excellent checks on the internal consistency of the scheme.

For deriving C_{24} we must evaluate its singular part

$$C_{24} \overset{\text{sing}}{\rightarrow} \frac{1}{4}\frac{1}{\bar{\varepsilon}}$$

from which we obtain

$$nC_{24} = [4 + (n-4)]\left[-\frac{1}{2}\frac{1}{n-4} + \mathcal{O}(1)\right] = -\frac{1}{2} + 4\, C_{24} + \mathcal{O}(n-4)$$

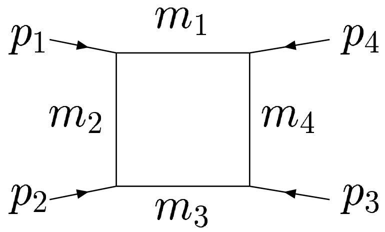
By inserting the previous results in the equation (*) we arrive at

$$C_{24} = \frac{1}{4} - \frac{m_1^2}{2} C_0 + \frac{1}{4}\left(C_0^{(0)} - f_1^c C_{11} - f_2^c C_{12}\right)$$

So, all C_{ij} are determined.

Third rank tensor is also needed and its reduction may be developped.

Four-point integrals, D-functions



The four-point functions are again much more complicated than the previous ones; only some particular cases will be considered.

The scalar four-point integral, D_0-function

$$i\pi^2 D_0\left(p_1^2, p_2^2, p_3^2, p_4^2, (p_1+p_2)^2, (p_2+p_3)^2; m_1, m_2, m_3, m_4\right)$$
$$= \mu^{4-n}\int d^n q \frac{1}{d_0 d_1 d_2 d_3}$$

the in this case are

$d_0 = q^2 + m_1^2 - i\epsilon,$ $\quad d_1 = (q+p_1)^2 + m_2^2 - i\epsilon$

$d_2 = (q+p_1+p_2)^2 + m_3^2 - i\epsilon,$ $\quad d_3 = (q+p_1+p_2+p_3)^2 + m_4^2 - i\epsilon$

with all four-momenta flowing inwards as shown in Fig, so that $p_1+p_2+p_3+p_4 = 0$. In terms of Feynman variables x, y and z

$$D_0 = \int_0^1 dx \int_0^x dy \int_0^y dz$$
$$\left(ax^2 + by^2 + gz^2 + cxy + hxz + jyz + dx + ey + kz + f\right)^{-2}$$

with

$$a = -p_{23}^2 = -p_3^2\,, \qquad b = -p_{12}^2 = -p_2^2$$

$$g = -p_{01}^2 = -p_1^2\,, \qquad c = -p_{13}^2 + p_{12}^2 + p_{23}^2$$

$$h = -p_{03}^2 - p_{12}^2 + p_{02}^2 + p_{13}^2\,, \qquad j = -p_{02}^2 + p_{01}^2 + p_{12}^2$$

$$d = m_3^2 - m_4^2 + p_{23}^2\,, \qquad e = m_2^2 - m_3^2 + p_{13}^2 - p_{23}^2$$

$$k = m_1^2 - m_2^2 + p_{03}^2 - p_{13}^2\,, \qquad f = m_4^2 - i\epsilon$$

and $p_{ij}^2 = (p_i - p_j)^2$

Reduction of tensor four-point integrals

$$i\pi^2\{D_\mu; D_{\mu\nu}; D_{\mu\nu\alpha}; D_{\mu\nu\alpha\beta}\} = \mu^{4-n}\int d^nq\frac{\{q_\mu; q_\mu q_\nu; q_\mu q_\nu q_\alpha; q_\mu q_\nu q_\alpha q_\beta\}}{d_0 d_1 d_2 d_3}$$

The tensor functions D_k, with $k = 11, 12, 13, 21, \ldots$

$$\begin{aligned}
D_\mu &= D_{11}p_{1\mu} + D_{12}p_{2\mu} + D_{13}p_{3\mu} \\
D_{\mu\nu} &= D_{21}p_{1\mu}p_{1\nu} + D_{22}p_{2\mu}p_{2\nu} + D_{23}p_{3\mu}p_{3\nu} \\
&\quad + D_{24}\{p_1p_2\}_{\mu\nu} + D_{25}\{p_1p_3\}_{\mu\nu} + D_{26}\{p_2p_3\}_{\mu\nu} + D_{27}\delta_{\mu\nu} \\
D_{\mu\nu\alpha} &= D_{31}p_{1\mu}p_{1\nu}p_{1\alpha} + D_{32}p_{2\mu}p_{2\nu}p_{2\alpha} + D_{33}p_{3\mu}p_{3\nu}p_{3\alpha} \\
&\quad + D_{34}\{p_2p_1p_1\}_{\mu\nu\alpha} + D_{35}\{p_3p_1p_1\}_{\mu\nu\alpha} + D_{36}\{p_1p_2p_2\}_{\mu\nu\alpha} \\
&\quad + D_{37}\{p_1p_3p_3\}_{\mu\nu\alpha} + D_{38}\{p_3p_2p_2\}_{\mu\nu\alpha} + D_{39}\{p_2p_3p_3\}_{\mu\nu\alpha} \\
&\quad + D_{310}\{p_1p_2p_3\}_{\mu\nu\alpha} + D_{311}\{p_1\delta\}_{\mu\nu\alpha} + D_{312}\{p_2\delta\}_{\mu\nu\alpha} \\
&\quad + D_{313}\{p_3\delta\}_{\mu\nu\alpha}
\end{aligned}$$

for rank-3 tensor one needs an additional symmetrized structure

$$\{pkl\}_{\mu\nu\alpha} = p_\mu\{kl\}_{\nu\alpha} + k_\mu\{pl\}_{\nu\alpha} + l_\mu\{pk\}_{\nu\alpha}$$

The reduction is performed by making use of the following identities:

$$\begin{aligned}
p_i\cdot q &= \frac{1}{2}\left(d_i - d_{i-1} + f_i^d\right) \\
f_1^d &= m_1^2 - m_2^2 - p_1^2, \qquad f_2^d = m_2^2 - m_3^2 + p_1^2 - Q^2 \\
f_3^d &= m_3^2 - m_4^2 - p_4^2 + Q^2
\end{aligned}$$

and the matrix X_4, given by $X_{4,ij} = p_i\cdot p_j$. The corresponding Gram determinant will be $\det X_4 = -\Delta_4$. The solution for D_{1i} is obtained by using the inverse matrix X_4^{-1} and $R^{(1)}_{123}$

$$\begin{pmatrix} D_{11} \\ D_{12} \\ D_{13}\end{pmatrix} = X_4^{-1}R^{(1)}_{123}\,, \qquad R^{(1)}_{123} = \frac{1}{2}\begin{pmatrix} D_0^{(1)} - D_0^{(0)} + f_1^d D_0 \\ D_0^{(2)} - D_0^{(1)} + f_2^d D_0 \\ D_0^{(3)} - D_0^{(2)} + f_3^d D_0\end{pmatrix}$$

where the pinches

$$\begin{aligned}
D_0^{(0)} &= C_0\left(p_2^2, p_3^2, P^2; m_2, m_3, m_4\right) \\
D_0^{(1)} &= C_0\left(p_3^2, p_4^2, Q^2; m_3, m_4, m_1\right) \\
D_0^{(2)} &= C_0\left(p_4^2, p_1^2, P^2; m_4, m_1, m_2\right) \\
D_0^{(3)} &= C_0\left(p_1^2, p_2^2, Q^2; m_1, m_2, m_3\right)
\end{aligned}$$

For the D_{2i} form factors the vectors $R_i^{(2)}$ are

$$R_{145}^{(2)} = \frac{1}{2}\begin{pmatrix} D_{11}^{(1)} + D_0^{(0)} + f_1^d D_{11} - 2D_{27} \\ D_{11}^{(2)} - D_{11}^{(1)} + f_2^d D_{11} \\ D_{11}^{(3)} - D_{11}^{(2)} + f_3^d D_{11} \end{pmatrix}$$

$$R_{426}^{(2)} = \frac{1}{2}\begin{pmatrix} D_{11}^{(1)} - D_{11}^{(0)} + f_1^d D_{12} \\ D_{12}^{(2)} - D_{11}^{(1)} + f_2^d D_{12} - 2D_{27} \\ D_{12}^{(3)} - D_{12}^{(2)} + f_3^d D_{12} \end{pmatrix}$$

$$R_{563}^{(2)} = \frac{1}{2}\begin{pmatrix} D_{12}^{(1)} - D_{12}^{(0)} + f_1^d D_{13} \\ D_{12}^{(2)} - D_{12}^{(1)} + f_2^d D_{13} \\ -D_{12}^{(2)} + f_3^d D_{13} - 2D_{27} \end{pmatrix}$$

with more pinches

$$\begin{aligned}
D_{11}^{(0)} &= C_{11}\left(p_2^2, p_3^2, P^2; m_2, m_3, m_4\right) \\
D_{11}^{(1)} &= -C_0\left(p_3^2, p_4^2, Q^2; m_3, m_4, m_1\right) - C_{12}\left(p_3^2, p_4^2, Q^2; m_3, m_4, m_1\right) \\
D_{11}^{(2)} &= -C_0\left(p_4^2, p_1^2, P^2; m_4, m_1, m_2\right) - C_{11}\left(p_4^2, p_1^2, P^2; m_4, m_1, m_2\right) \\
&\quad + C_{12}\left(p_4^2, p_1^2, P^2; m_4, m_1, m_2\right) \\
D_{11}^{(3)} &= C_{11}\left(p_1^2, p_2^2, Q^2; m_1, m_2, m_3\right) \\
D_{12}^{(0)} &= C_{12}\left(p_2^2, p_3^2, P^2; m_2, m_3, m_4\right) \\
D_{12}^{(1)} &= C_{11}\left(p_3^2, p_4^2, Q^2; m_3, m_4, m_1\right) - C_{12}\left(p_3^2, p_4^2, Q^2; m_3, m_4, m_1\right) \\
D_{12}^{(2)} &= -C_0\left(p_4^2, p_1^2, P^2; m_4, m_1, m_2\right) - C_{11}\left(p_4^2, p_1^2, P^2; m_4, m_1, m_2\right) \\
D_{12}^{(3)} &= C_{12}\left(p_1^2, p_2^2, Q^2; m_1, m_2, m_3\right)
\end{aligned}$$

In above equations, $Q^2 = (p_1 + p_2)^2$, and $P^2 = (p_2 + p_3)^2$,
for *direct* boxes, while
$P^2 = (p_2 + p_4)^2$, and $p_3^2 \leftrightarrow p_4^2$,
for the *crossed* ones.

We present the solution for the direct boxes only. The solution for the crossed boxes is given by the interchange $p_3^2 \leftrightarrow p_4^2$. The D_{27} form factor is immediately given in terms of lower rank integrals

$$D_{27} = -m_1^2 D_0 + \frac{1}{2}\left[D_0^{(0)} - f_1^d D_{11} - f_2^d D_{12} - f_3^d D_{13}\right]$$

The other D_2 form factors

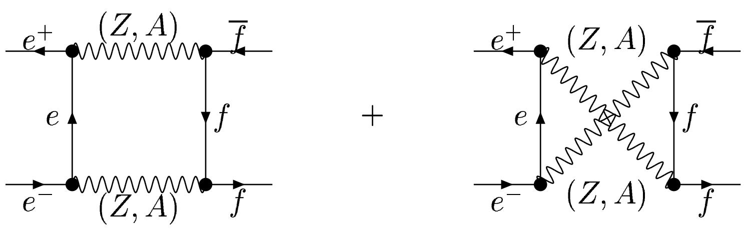
$$\begin{pmatrix} D_{2i} \\ D_{2j} \\ D_{2l} \end{pmatrix} = X_4^{-1} R_{ijl}^{(2)}$$

with $\{ijl\} = \{145\}, \{426\}$ and $\{563\}$.

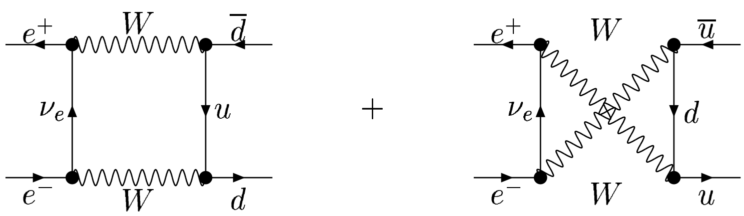
Minimal SM boxes for the e^+e^- annihilation into fermion pairs can be *direct* or *crossed*. For WW internal lines there is a peculiar aspect due to charge conservation:

direct box only	for	$e^+e^- \to d\overline{d}$
crossed box only	for	$e^+e^- \to u\overline{u}$

The box diagrams:



AA-, ZA-, ZZ-boxes



WW-boxes

Some particular cases of D_0-functions

Case 1).
The most general expression one encounters in considering ZZ and WW boxes

$$\begin{aligned} p_i^2 &= 0\,, \qquad (p_1+p_2)^2 = Q^2, \qquad (p_2+p_3)^2 = P^2 \\ m_1 &= M_1\,, \quad m_2 = 0\,, \quad m_3 = M_1\,, \quad m_4 = M_2 \end{aligned}$$

With an appropriate choice of Feynman parameters it may be presented and calculated as follows:

$$\begin{aligned} D_0\left(0,0,0,0,Q^2,P^2;M_1,0,M_1,M_2\right) &= \int_0^1 dz \int_0^1 y dy \int_0^1 dx \\ &\times \frac{1}{\left[M_1^2 y + M_2^2(1-y) + P^2(1-y)(1-z) + Q^2 z y^2 x(1-x)\right]^2} \\ = \frac{1}{Q^2(P^2+M_2^2)\sqrt{d_4}} \sum_{i=1}^{4}\sum_{j=1}^{2} & (-1)^{\delta_{i3}+\delta_{j2}} \left[\mathrm{Li}_2\left(\frac{\bar{x}_j}{\bar{x}_j - x_i}\right) - \mathrm{Li}_2\left(\frac{\bar{x}_j - 1}{\bar{x}_j - x_i}\right)\right] \end{aligned}$$

with the six roots

$$\begin{aligned} x_{1,2} &= \frac{1}{2}\left(1 \mp \sqrt{1+\frac{4M_1^2}{Q^2}}\right), \qquad & \bar{x}_{1,2} &= \frac{x_4}{2}\left(1 \mp \sqrt{d_4}\right) \\ x_3 &= \frac{M_2^2}{M_2^2 - M_1^2}\,, & x_4 &= \frac{P^2+M_2^2}{P^2+M_2^2-M_1^2} \end{aligned}$$

and

$$d_4 = 1 + \frac{4M_1^2 P^2\left(P^2+M_2^2-M_1^2\right)}{Q^2\left(P^2+M_2^2\right)^2}$$

For $M_2 = 0$ (in practical applications $m_t = 0$), it simplifies to

$$D_0\left(0,0,0,0,Q^2,P^2;M_1,0,M_1,0\right) = \frac{2}{Q^2 P^2 \sqrt{d_4^{(0)}}} \sum_{ij=1}^{2} (-1)^{i+1}\, \mathrm{Li}_2\left(\frac{\tilde{x}_i}{\tilde{x}_i - x_j}\right)$$

with the other roots

$$\tilde{x}_{1,2} = \frac{x_4}{2}\left(1 \mp \sqrt{d_4^{(0)}}\right), \qquad d_4^{(0)} = 1 + \frac{4M_1^2\left(P^2 - M_1^2\right)}{Q^2 P^2}$$

Case 2). One encounters this case when considering ZA and AA boxes where we introduce three auxiliary integrals

$$i\pi^2 \bar{J}_{\gamma\gamma}\left(Q^2, P^2; m_e, m_f\right) = \mu^{4-n} \int d^n q \frac{2q\cdot(q+Q)}{d_0(0)\, d_1(m_e)\, d_2(0)\, d_3(m_f)}$$

$$i\pi^2 \bar{J}_{\gamma Z}\left(Q^2, P^2; m_e, m_f\right) = \mu^{4-n} \int d^n q \frac{2q\cdot Q}{d_0(0)\, d_1(m_e)\, d_2(M_Z)\, d_3(m_f)}$$

$$i\pi^2 \bar{J}_{Z\gamma}\left(Q^2, P^2; m_e, m_f\right) = \mu^{4-n} \int d^n q \frac{2Q\cdot(q+Q)}{d_0(M_Z)\, d_1(m_e)\, d_2(0)\, d_3(m_f)}$$

which are simple *both* to calculate *and* to reduce to scalar functions D_0

$$\begin{aligned}
&D_0\left(-m_e^2, -m_e^2, -m_f^2, -m_f^2, Q^2, P^2; 0, m_e, 0, m_f\right)\\
&= \frac{1}{Q^2}\Big[-\bar{J}_{\gamma\gamma}\left(Q^2, P^2; m_e, m_f\right)\\
&+C_0\left(-m_e^2, -m_f^2, P^2; m_e, 0, m_f\right) + C_0\left(-m_f^2, -m_e^2, P^2; m_f, 0, m_e\right)\Big]\\
&D_0\left(-m_e^2, -m_e^2, -m_f^2, -m_f^2, Q^2, P^2; 0, m_e, M_Z, m_f\right)\\
&= \frac{1}{Q^2+M_Z^2}\Big[-\bar{J}_{\gamma Z}\left(Q^2, P^2; m_e, m_f\right)\\
&-C_0\left(-m_e^2, -m_f^2, P^2; m_e, M_Z, m_f\right) + C_0\left(-m_f^2, -m_e^2, P^2; m_f, 0, m_e\right)\Big]\\
&D_0\left(-m_e^2, -m_e^2, -m_f^2, -m_f^2, Q^2, P^2; M_Z, m_e, 0, m_f\right)\\
&= \frac{1}{Q^2+M_Z^2}\Big[\bar{J}_{Z\gamma}\left(Q^2, P^2; m_e, m_f\right)\\
&+C_0\left(-m_e^2, -m_f^2, P^2; m_e, 0, m_f\right) - C_0\left(-m_f^2, -m_e^2, P^2; m_f, M_Z, m_e\right)\Big]
\end{aligned}$$

The answers for the auxiliary integrals in terms of one fold integrals

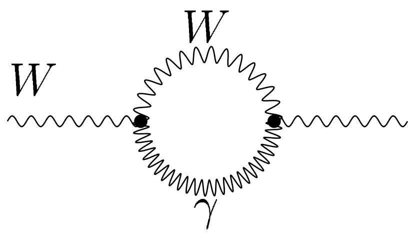
$$\bar{J}_{\gamma\gamma}\left(Q^2, P^2; m_e, m_f\right) = \int_0^1 dx \frac{1}{\chi\left(P^2; m_e, m_f\right)} \ln\frac{\chi\left(P^2; m_e, m_f\right)}{Q^2}$$

$$\bar{J}_{\gamma Z}\left(Q^2, P^2; m_e, m_f\right) = -\bar{J}_{Z\gamma}\left(Q^2, P^2; m_e, m_f\right) = \ln\frac{M_Z^2+Q^2}{M_Z^2}\int_0^1 dx \frac{1}{\chi}$$

Here $\chi\left(P^2; m_e, m_f\right) = P^2 x(1-x) + m_e^2(1-x) + m_f^2 x$ is the usual quadratic form.

Special PV functions: $a, b, c^{(j)}, d^{(j)}$

Passarino-Veltman (PV) functions, A, B, C, D, are sufficient to calculate one-loop corrections in $\xi = 1$ and U-gauges. In the R_ξ-gauge additional complications arise. Consider diagrams with an internal photonic lines with photon propagators which contain an additional term $\left(\xi_A^2 - 1\right) q_\mu q_\nu / q^2$. This leads to a special class of two- (three-, four-) point functions.



The scalar b_0-function

$$i\pi^2\, b_0\left(p^2; m\right) = \mu^{4-n} \int d^n q \frac{1}{\left(q^2\right)^2 \left(\left(q+p\right)^2 + m^2\right)}$$

This is badly divergent object in the infrared regime.

$$n = 4 + \varepsilon', \qquad \varepsilon' > 0$$

With $\chi = 1 + (1-x)p^2/m^2$

$$\begin{aligned} b_0\left(p^2; m\right) &= \pi^{\varepsilon'/2}\Gamma\left(1 - \frac{\varepsilon'}{2}\right)\left(\frac{m^2}{\mu^2}\right)^{\varepsilon'/2} \int_0^1 dx\, x^{-1+\varepsilon'/2} \frac{(1-x)\chi^{-1+\varepsilon'/2}}{m^2} \\ &\approx \pi^{\varepsilon'/2}\Gamma\left(1 - \frac{\varepsilon'}{2}\right)\left(\frac{m^2}{\mu^2}\right)^{\varepsilon'/2} \int_0^1 dx\, x^{-1+\varepsilon'/2}\, h(x) \\ h(x) &= \frac{1-x}{m^2\chi}\left(1 + \frac{\varepsilon'}{2}\ln\chi\right) \end{aligned}$$

By adding and subtracting $h(0)$ and by noticing that $x^{-1}\left[h(x) - h(0)\right]$ is finite for $\varepsilon' \to 0$ we obtain

$$b_0\left(p^2; m\right) = \frac{1}{p^2 + m^2}\left[\frac{1}{\hat{\varepsilon}} + \ln\frac{m^2}{\mu^2} + \left(1 - \frac{m^2}{p^2}\right)\ln\left(1 + \frac{p^2}{m^2}\right)\right]$$

By analytical continuation this integral is now defined in the whole n-plane and it shows an infrared pole at $n = 4$.

Vector integral

$$i\pi^2 b_1\left(p^2;m\right)\,p_\mu \;=\; \mu^{4-n}\int d^n q\frac{q_\mu}{\left(q^2\right)^2\left(\left(q+p\right)^2+m^2\right)}$$
$$\;=\; -i\pi^{n/2}p_\mu\Gamma\left(3-\frac{n}{2}\right)\int_0^1 dx\frac{x^{n/2-2}\left(1-x\right)}{\left(m^2\chi\right)^{3-n/2}}$$

Thus this function is free of singularities

$$b_1\left(p^2;m\right)=-\int_0^1 dx\frac{x}{m^2+p^2x}=-\frac{1}{p^2}\left[1-\frac{m^2}{p^2}\ln\left(1+\frac{p^2}{m^2}\right)\right]$$

We have an alternative way of evaluating $b_1\left(p^2;m\right)$.
With $d=\left(q+p\right)^2+m^2$, we derive

$$i\pi^2p^2b_1\left(p^2;m\right) \;=\; \frac{1}{2}\int d^n q\left[\frac{1}{\left(q^2\right)^2}-\frac{1}{q^2d}-\frac{p^2+m^2}{\left(q^2\right)^2d}\right]$$
$$p^2b_1\left(p^2;m\right) \;=\; \frac{1}{2}\left[a_0-B_0\left(p^2;0,m\right)-\left(p^2+m^2\right)\,b_0\left(p^2;m\right)\right]$$

In the previous derivation we have introduced a new integral,

$$i\pi^2\,a_0=\int d^n q\frac{1}{\left(q^2\right)^2}$$

which deserves a careful examination. Since one has

$$B_0\left(p^2;0,m\right)=\frac{1}{\bar{\varepsilon}}+2-\ln\frac{m^2}{\mu^2}-\left(1+\frac{m^2}{p^2}\right)\ln\left(1+\frac{p^2}{m^2}\right)$$

from above equations which are valid for any n, we obtain the proper definition of this integral

$$a_0=\frac{1}{\hat{\varepsilon}}+\frac{1}{\bar{\varepsilon}}=0$$

In this way we derive a typical relation between BP and PV function

$$p^2b_1\left(p^2;m\right)=-\frac{1}{2}\left[B_0\left(p^2;0,m\right)+\left(p^2+m^2\right)\,b_0\left(p^2;m\right)\right]$$

The rank two tensor integral

$$i\pi^2\left(b_{21}p_\mu p_\nu+b_{22}\delta_{\mu\nu}\right)=\mu^{4-n}\int d^nq\frac{q_\mu q_\nu}{\left(q^2\right)^2\left(\left(q+p\right)^2+m^2\right)}$$

A direct calculation shows that

$$b_{22}\left(p^2;m\right)=\frac{1}{2}\pi^{n/2-2}\mu^{4-n}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 dx\,x\left(1-x\right)^{n/2-2}\left(m^2+p^2x\right)^{n/2-2}$$

This means

$$n\,b_{22}\left(p^2;m\right)=4\,b_{22}\left(p^2;m\right)-\frac{1}{2}$$

By application of the usual method one gets the system

$$\begin{aligned}
p^2b_{21}\left(p^2;m\right)+nb_{22}\left(p^2;m\right) &= B_0\left(p^2;0,m\right),\\
p^2b_{21}\left(p^2;m\right)+b_{22}\left(p^2;m\right) &= \frac{1}{2}\left[B_1\left(p^2;0,m\right)+\left(p^2+m^2\right)b_1\left(p^2;m\right)\right]
\end{aligned}$$

and its solution

$$\begin{aligned}
b_{22}\left(p^2;m\right) &= \frac{1}{3}B_0\left(p^2;0,m\right)+\frac{1}{6}\left[B_1\left(p^2;0,m\right)+\left(p^2+m^2\right)\,b_1\left(p^2;m\right)+1\right]\\
b_{21}\left(p^2;m\right) &= -4\,b_{22}\left(p^2;m\right)+B_0\left(p^2;0,m\right)+\frac{1}{2}
\end{aligned}$$

After the identification $1/\hat{\varepsilon}=-1/\bar{\varepsilon}$ the following identities hold:

$$\begin{aligned}
\left(p^2+m^2\right)^2b_0\left(p^2;m\right) &= 2A_0\left(m\right)+2\,p^2-\left(p^2-m^2\right)B_0\left(p^2;0,m\right)\\
\left(p^2+m^2\right)b_1\left(p^2;m\right) &= -1-\frac{1}{p^2}\left[A_0\left(m\right)+m^2\,B_0\left(p^2;0,m\right)\right]
\end{aligned}$$

which give *relations* between BP and PV functions

$$\begin{aligned}
b_{22}\left(p^2;m\right) &= \frac{1}{3}B_0\left(p^2;0,m\right)\\
&\quad+\frac{1}{6}\left\{B_1\left(p^2;0,m\right)-\frac{1}{p^2}\left[A_0\left(m\right)+m^2B_0\left(p^2;0,m\right)\right]\right\}\\
b_{21}\left(p^2;m\right) &= \frac{1}{p^2}\left\{-\frac{1}{3}B_0\left(p^2;0,m\right)\right.\\
&\quad\left.-\frac{2}{3}\left(B_1\left(p^2;0,m\right)-\frac{1}{p^2}\left[A_0\left(m\right)+m^2B_0\left(p^2;0,m\right)\right]\right)+\frac{1}{2}\right\}
\end{aligned}$$

One more BP-series

The scalar function $\hat{b}_0$

$$i\pi^2\hat{b}_0\left(Q^2\right) = \mu^{4-n} \int d^n q \frac{1}{\left(q^2\right)^2\left((q+Q)^2\right)^2}$$

With $n = 4 + \varepsilon'$, a straightforward calculation leads to

$$\begin{aligned}
\hat{b}_0\left(Q^2\right) &= \frac{1}{\left(Q^2\right)}\pi^{\varepsilon'/2}\Gamma\left(2-\frac{\varepsilon'}{2}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon'/2} \int_0^1 dx \left[x\left(1-x\right)\right]^{-1+\varepsilon'/2} \\
&= \frac{2}{\left(Q^2\right)^2}\pi^{\varepsilon'/2}\Gamma\left(2-\frac{\varepsilon'}{2}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon'/2} \mathrm{B}\left(\frac{\varepsilon'}{2}, 1+\frac{\varepsilon'}{2}\right) \\
&= \frac{2}{\left(Q^2\right)^2}\left(\frac{1}{\hat{\varepsilon}} + \ln\frac{m^2}{\mu^2} - 1\right) \equiv \frac{2}{Q^2}\left[b_0\left(Q;0\right) - \frac{1}{Q^2}\right]
\end{aligned}$$

Vector and tensor integrals

$$\begin{aligned}
i\pi^2\hat{b}_1\left(Q^2\right)Q_\mu &= \mu^{4-n}\int d^n q \frac{q_\mu}{\left(q^2\right)^2\left((q+Q)^2\right)^2} \\
i\pi^2\left[\hat{b}_{21}\left(Q^2\right)Q_\mu Q_\nu + \hat{b}_{22}\left(Q^2\right)\delta_{\mu\nu}\right] &= \mu^{4-n}\int d^n q \frac{q_\mu q_\nu}{\left(q^2\right)^2\left((q+Q)^2\right)^2}
\end{aligned}$$

By a straightforward calculation one obtains

$$\begin{aligned}
\hat{b}_1\left(Q^2\right) &= -\frac{1}{2}\hat{b}_0\left(Q^2\right) \\
\hat{b}_{21}\left(Q^2\right) &= \frac{1}{Q^2}\left[b_0\left(Q^2;0\right) - \frac{2}{Q^2}\right] \\
\hat{b}_{22}\left(Q^2\right) &= \frac{1}{2Q^2}
\end{aligned}$$

Actually only infrared finite objects will appear in the calculation, like for instance

$$\int d^n q \frac{q_\mu\left(q+Q\right)_\nu}{\left(q^2\right)^2\left((q+Q)^2\right)^2} = \frac{1}{2Q^2}\delta_{\mu\nu} - \frac{1}{\left(Q^2\right)^2}Q_\mu Q_\nu$$

The full collection of scalar, vectors and tensors is, nevertheless, needed if one wants to develop an automatic computer program for generation and calculation of one-loop diagrams.

$c_i^{(j)}$-functions

In considering arbitrary four fermion processes one encounters additional structures. An example is given by four classes of special functions, to be termed $c_i^{(j)}$ functions, $j = 0, 1, 2, 02$. The function with $j = 02$ is a pinch of the $\gamma\gamma$-box diagram. They are defined by the following equations

$$
\begin{aligned}
i\pi^2 c^{(0)}_{\{1,\mu,\mu\nu\}}\left(p_1^2, p_2^2, Q^2; 0, m_2, m_3\right) &= \mu^{4-n} \int d^n q \, \frac{\{1, q_\mu, q_\mu q_\nu\}}{d_0^2 d_1 d_2} \\
i\pi^2 c^{(1)}_{\{1,\mu,\mu\nu\}}\left(p_1^2, p_2^2, Q^2; m_1, 0, m_3\right) &= \mu^{4-n} \int d^n q \, \frac{\{1, q_\mu, q_\mu q_\nu\}}{d_0 d_1^2 d_2} \\
i\pi^2 c^{(2)}_{\{1,\mu,\mu\nu\}}\left(p_1^2, p_2^2, Q^2; m_1, m_2, 0\right) &= \mu^{4-n} \int d^n q \, \frac{\{1, q_\mu, q_\mu q_\nu\}}{d_0 d_1 d_2^2} \\
i\pi^2 c^{(02)}_{\{1,\mu,\mu\nu\}}\left(p_1^2, p_2^2, Q^2; 0, m_2, 0\right) &= \mu^{4-n} \int d^n q \, \frac{\{1, q_\mu, q_\mu q_\nu\}}{d_0^2 d_1 d_2^2}
\end{aligned}
$$

$d_i^{(j)}$-functions

Finally there will be special functions associated with four point integrals. The class of the $d_i^{(j)}$-functions is reacher than the one of the $c_i^{(j)}$-functions. As usual, we limit ourself in the study of those functions which appear in the actual consideration of four-fermion processes. This is why only three classes of $d_i^{(j)}$-functions with $j = 0, 2, 02$ are considered:

$$
\begin{aligned}
i\pi^2 d^{(0)}_{\{1,\mu,\mu\nu\}}\left(p_1^2, p_2^2, p_3^2, p_4^2, Q^2, P^2; 0, m_2, m_3, m_4\right) &= \mu^{4-n} \int d^n q \, \frac{\{1, q_\mu, q_\mu q_\nu\}}{d_0^2 d_1 d_2 d_3} \\
i\pi^2 d^{(2)}_{\{1,\mu,\mu\nu\}}\left(p_1^2, p_2^2, p_3^2, p_4^2, Q^2, P^2; m_1, m_2, 0, m_4\right) &= \mu^{4-n} \int d^n q \, \frac{\{1, q_\mu, q_\mu q_\nu\}}{d_0 d_1 d_2^2 d_3} \\
i\pi^2 d^{(02)}_{\{1,\mu,\mu\nu\}}\left(p_1^2, p_2^2, p_3^2, p_4^2, Q^2, P^2; 0, m_2, 0, m_4\right) &= \mu^{4-n} \int d^n q \, \frac{\{1, q_\mu, q_\mu q_\nu\}}{d_0^2 d_1 d_2^2 d_3}
\end{aligned}
$$

Only reduction is needed. The scalar BP-functions do not appear in $2 \to 2$ on-mass-shell processes.

Summary of Level 2

Standard Model, its Fields and Lagrangian

Different gauges: $R_\xi, \quad \xi = 1, \quad U$
Gauge Invariance

Feynman Rules, *Building* of Diagrams

Dimension regularization

N-point functions

Calculation of integrals:
$A,\ B,\ C,\ D,\ a,\ b,\ c,\ d$ -functions

Time to *calculate* diagrams

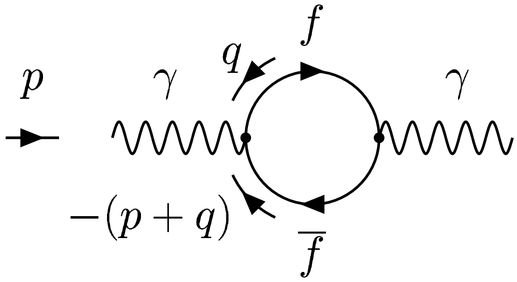
Ultraviolet and Infrared divergences

$$
\begin{aligned}
n &= 4 - \varepsilon \quad \to \quad \frac{1}{\bar{\varepsilon}} = -\frac{2}{n-4} - \gamma - \ln \pi \\
n &= 4 + \varepsilon' \quad \to \quad \frac{1}{\hat{\varepsilon}} = +\frac{2}{n-4} + \gamma + \ln \pi
\end{aligned}
$$

$$
\frac{1}{\bar{\varepsilon}} + \frac{1}{\hat{\varepsilon}} = 0
$$

Calculation of photonic self-energy diagram

The photon self-energy is described by a tensor, $\Pi_{\mu\nu}$



Applying Feynman rules for vertices and propagators

$$\Pi_{\mu\nu} = e^2 Q_e^2 \int d^n q \frac{Tr[(i\not{q} + m_f)\gamma_\mu (i\not{p} + i\not{q} + m_f)\gamma_\nu]}{\left(q^2 + m_f^2\right)\left[(q+p)^2 + m_f^2\right]}$$

$$= 4e^2 Q_e^2 \int d^n q \frac{\delta_{\mu\nu}\left(q^2 + m_f^2 + qp\right) - (q_\mu p_\nu + q_\nu p_\mu) - 2q_\mu q_\nu}{\left(q^2 + m_f^2\right)\left[(q+p)^2 + m_f^2\right]}$$

Using definitions of A and B functions we immediately get the answer

$$\Pi_{\mu\nu} = i\pi^2\, 4e^2 Q_e^2 \Big\{ \delta_{\mu\nu}\left[A_0\left(m_f^2\right) + p^2 B_1\left(p^2; m_f, m_f\right)\right]$$
$$-2p_\mu p_\nu B_1\left(p^2; m_f, m_f\right)$$
$$-2\left[B_{22}\left(p^2; m_f, m_f\right)\delta_{\mu\nu} + B_{21}\left(p^2; m_f, m_f\right) p_\mu p_\nu\right]\Big\}$$
$$= i\pi^2\, 4e^2 Q_e^2 \Big\{ \delta_{\mu\nu}\left[A_0\left(m_f\right) + p^2 B_1\left(p^2; m_f, m_f\right) - 2B_{22}\left(p^2; m_f, m_f\right)\right]$$
$$-2p_\mu p_\nu\left[B_1\left(p^2; m_f, m_f\right) + B_{21}\left(p^2; m_f, m_f\right)\right]\Big\}$$

It must be *transverse* as consequence of QED $U(1)$ gauge invariance:

$$\Pi_{\mu\nu} = i\pi^2\, 4e^2 Q_e^2 \left(p^2 \delta_{\mu\nu} - p_\mu p_\nu\right) \Pi\left(p^2\right)$$

This property will be satisfied if

$$A_0\left(m_f\right) + p^2 B_1\left(p^2; m_f, m_f\right) - 2B_{22}\left(p^2; m_f, m_f\right)$$
$$= 2p^2\left[B_1\left(p^2; m_f, m_f\right) + B_{21}\left(p^2; m_f, m_f\right)\right]$$

Four functions: $A_0,\ B_1,\ B_{21},\ B_{22}$, maybe *reduced* to only two scalar integrals, $A_0,\ B_0$. Therefore, relations are possible. Indeed, from general result, for equal masses one has:

$$
\begin{aligned}
B_1\left(p^2;m,m\right) &= -\frac{1}{2}B_0\left(p^2;m,m\right)\\
B_{21}\left(p^2;m,m\right) &= \frac{6m^2+p^2}{18p^2}+\frac{1}{3p^2}A_0\left(m\right)+\frac{p^2+m^2}{3p^2}B_0\left(p^2;m,m\right)\\
B_{22}\left(p^2;m,m\right) &= -\frac{6m^2+p^2}{18}+\frac{1}{6}A_0\left(m\right)-\frac{p^2+4m^2}{6}B_0\left(p^2;m,m\right)
\end{aligned}
$$

and the wanted equality is immediately verified.

Result for $\Pi\left(p^2\right)$:

$$
\begin{aligned}
\Pi\left(p^2\right) &= 2\left[B_{21}\left(p^2;m_f,m_f\right)+B_1\left(p^2;m_f,m_f\right)\right]\\
&= \frac{6m_f^2+p^2}{9p^2}+\frac{2}{3p^2}A_0\left(m_f\right)-\frac{p^2-2m_f^2}{3p^2}B_0\left(p^2;m_f,m_f\right)\\
&= -\frac{1}{3}\left(\frac{1}{\bar{\varepsilon}}-\ln\frac{m_f^2}{\mu^2}\right)+\frac{1}{9}+\frac{1}{3}\left(1-2\frac{m_f^2}{p^2}\right)\left[\beta\ln\frac{\beta+1}{\beta-1}-2\right]
\end{aligned}
$$

where

$$
\beta=\sqrt{1+4\frac{m_f^2}{p^2}}
$$

And two limiting cases:

$$
\begin{aligned}
&\text{for}\quad p^2\gg m_f^2 \qquad \Pi\left(p^2\right)=-\frac{1}{3}\left(-\frac{1}{\bar{\varepsilon}}\ln\frac{m_f^2}{\mu^2}\right)-\frac{5}{9}+\frac{1}{3}\ln\frac{p^2}{m_f^2}\\
&\text{for}\quad p^2\ll m_f^2 \qquad \Pi\left(p^2\right)=-\frac{1}{3}\left(\frac{1}{\bar{\varepsilon}}-\ln\frac{m_f^2}{\mu^2}\right)+\frac{p^2}{15m_f^2}+\cdots
\end{aligned}
$$

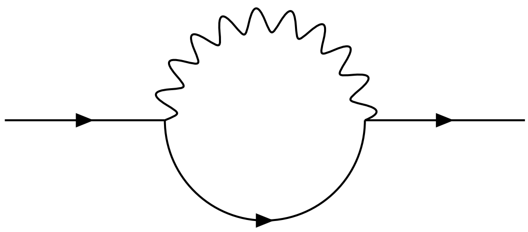
The finite difference

$$
\Pi(p^2)-\Pi(0)=-\frac{5}{9}+\frac{4m_f^2}{3p^2}+\frac{1}{3}\left(1-2\frac{m_f^2}{p^2}\right)\beta\ln\frac{\beta+1}{\beta-1}
$$

is renormalized photonic self-energy as will be proved later.

Fermionic self-energy

Fermionic self-energy is a 4×4 matrix:



Applying Feynman rules

$$\begin{aligned}\Sigma(\not p) &= -e^2 Q_e^2 \int d^n q \frac{\gamma_\mu \left(i \not q + m_f \right) \gamma_\mu}{\left(q^2 + m_f^2 - i\epsilon\right)\left[(q+p)^2 - i\epsilon\right]} \\ &= i\pi^2 \left(-e^2 Q_e^2\right)\left[(2-n) B_1\left(p^2; m_f, 0\right) i\not p + n m_f B_0\left(p^2; m_f, 0\right)\right]\end{aligned}$$

Since

$$\frac{1}{\bar\varepsilon} = \frac{2}{4-n} + \text{finite terms} \quad \to \quad \frac{n}{\bar\varepsilon} = \frac{2n}{4-n} = \frac{4}{\bar\varepsilon} - 2$$

and

$$n B_0 = 4 B_0 - 2, \qquad n B_1 = 4 B_1 + 1$$

the final result for fermionic self energy

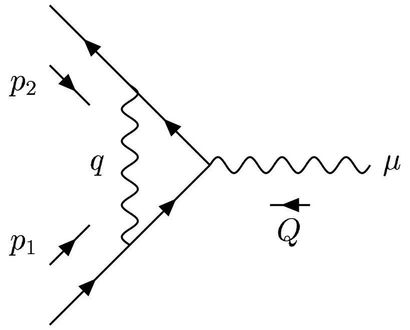
$$\Sigma(\not p) = i\pi^2\, e^2 Q_e^2 \left\{\left[2 B_1\left(p^2; m_f, 0\right) + 1\right] i\not p - m_f \left[4 B_0\left(p^2; m_f, 0\right) - 2\right]\right\}$$

The fermionic self-energy is well-defined on the fermion mass shell, but not its derivative, $\partial\Sigma(\not p)/\partial p^2|_{p^2=-m_f^2}$, which shows a singularity due to the zero mass of the photon. We remind

$$\begin{aligned}\frac{\partial}{\partial p^2} B_0\left(p^2; m, 0\right)\Big|_{p^2=-m^2} &= -\frac{1}{2m^2}\left(\frac{1}{\hat\varepsilon} - 2 + \ln\frac{m^2}{\mu^2}\right) \\ \frac{\partial}{\partial p^2} B_1\left(p^2; m, 0\right)\Big|_{p^2=-m^2} &= \frac{1}{2m^2}\left(\frac{1}{\hat\varepsilon} - 3 + \ln\frac{m^2}{\mu^2}\right)\end{aligned}$$

QED vertex

The one-loop QED $f\overline{f}\gamma$ vertex corresponds to the diagram



For on mass shell fermions the most general structure compatible with both Lorentz and gauge invariance:

$$\Lambda_\mu = (2\pi)^4 \, i \, ieQ_e \frac{e^2}{16\pi^2} \left[\gamma_\mu F_1 + \sigma_{\mu\nu} (p_1 + p_2)_\nu \, F_2\right]$$

Note:

- $(2\pi)^4 \, i \, ieQ_e\gamma_\mu \to (2\pi)^4 \, i \, ieQ_e\gamma_\mu + \Lambda_\mu \,, \quad Q_e = -1;$
- F_1 – the Dirac electric form factor; ultraviolet and infrared divergent;
- F_2 – the anomalous magnetic moment of the electron; it is finite.

For on-shell fermions: $\bar{v}(p_2)\, \not{p}_2 = -i\, m\bar{v}(p_2)\,, \not{p}_1\, u(p_1) = i\, mu(p_1)\,,$

$$p_1^2 = p_2^2 = -m^2 \,, \quad Q^2 = (p_1 + p_2)^2 = -2m^2 + 2p_1 \cdot p_2$$

$$\Lambda_\mu = i\,(eQ_e)^3 \mu^{4-n} \int d^n q \frac{1}{q^2\left[(q + p_1)^2 + m^2\right]\left[(q - p_2)^2 + m^2\right]} N_\mu$$

$$N_\mu = -4p_1 \cdot p_2\gamma_\mu + 2\,(\not{p}_1\gamma_\alpha\gamma_\mu - \gamma_\mu\gamma_\alpha\not{p}_2)\, q_\alpha + (2 - n)\,\gamma_\alpha\gamma_\mu\gamma_\beta q_\alpha q_\beta$$

With the standard Feynman parameterization, and notations:

$$k_x = xp_2 - (1 - x)\, p_1 \,, \qquad \chi\left(Q^2, x\right) = Q^2 x\,(1 - x) + m^2$$

$$\begin{aligned}\Lambda_\mu &= i\,(eQ_e)^3\, \Gamma(3) \int_0^1 dx \int_0^1 dyy\, \mu^{4-n} \int d^n q \frac{1}{(q^2 - 2\,y\,q\cdot k_x)^3} N_\mu \\ &= i\pi^2 i\,(eQ_e)^3 \left[-\left(Q^2 + 2m^2\right)\gamma_\mu S + 2\,(\not{p}_1\gamma_\alpha\gamma_\mu - \gamma_\mu\gamma_\alpha\not{p}_2)\, V_\alpha \right. \\ &\quad \left. +\gamma_\alpha\gamma_\mu\gamma_\beta T_{\alpha\beta}\right]\end{aligned}$$

For the scalar integral we use the *infrared* regulator ε':

$$\begin{aligned} S &= 2\Gamma(3)\frac{\mu^{-\varepsilon'}}{i\pi^2}\int_0^1 dx\int_0^1 dyy\int\frac{d^nq}{\left(q^2-2yq\cdot k_x\right)^3} \\ &= 2\pi^{\varepsilon'/2}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\int_0^1 dx\int_0^1 dyy^{-1+\varepsilon'}\frac{1}{\chi(Q^2,x)}\left[\frac{\chi(Q^2,x)}{\mu^2}\right]^{\varepsilon'/2} \end{aligned}$$

y-integration can be performed for any value of n

$$\int_0^1 dyy^{-k-\varepsilon}=\frac{1}{1-k-\varepsilon},\qquad k=-1,0,1...$$

$$S = 2\frac{\pi^{\varepsilon'/2}}{\varepsilon'}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\int_0^1 dx\frac{1}{\chi(Q^2,x)}\left[\frac{\chi(Q^2,x)}{\mu^2}\right]^{\varepsilon'/2}$$

expanding around $\varepsilon'=0$, we arrive at some expression in terms of a one-fold integral:

$$S=\int_0^1 dx\frac{1}{\chi(Q^2,x)}\left[\frac{1}{\hat{\varepsilon}}+\ln\frac{\chi(Q^2,x)}{\mu^2}\right]$$

For the vector and tensor we use the *ultraviolet* regulator ε.

$$\begin{aligned} V_\alpha &= \Gamma(3)\frac{\mu^{\varepsilon}}{i\pi^2}\int_0^1 dx\int_0^1 dyy\int\frac{d^nqq_\alpha}{\left(q^2-2yq\cdot k_x\right)^3} \\ &= \pi^{-\varepsilon/2}\Gamma\left(1+\frac{\varepsilon}{2}\right)\int_0^1 dxk_{x,\alpha}\int_0^1 dyy^{-\varepsilon}\frac{1}{\chi(Q^2,x)}\left[\frac{\chi(Q^2,x)}{\mu^2}\right]^{-\varepsilon/2} \\ &= \frac{(p_2-p_1)_\alpha}{2}\pi^{-\varepsilon/2}\frac{\Gamma(1+\varepsilon/2)}{1-\epsilon}\int_0^1 dx\frac{1}{\chi(Q^2,x)}\left[\frac{\chi(Q^2,x)}{\mu^2}\right]^{-\varepsilon/2} \end{aligned}$$

Vector is finite and we may set $\varepsilon=0$

$$V_\alpha = \frac{(p_2-p_1)_\alpha}{2}F_2,\qquad F_2=\int_0^1 dx\frac{1}{\chi(Q^2,x)}$$

Dirac algebra for vector

$$2\left(\not{p}_1\gamma_\alpha\gamma_\mu-\gamma_\mu\gamma_\alpha\not{p}_2\right)\frac{(p_2-p_1)_\alpha}{2}=2\left[\left(Q^2+4m^2\right)\gamma_\mu+im\left(p_1-p_2\right)_\mu\right]$$

For the tensor integral we have to consider full contraction

$$\begin{aligned}
\gamma_\alpha\gamma_\mu\gamma_\beta T_{\alpha\beta} &= \Gamma(3)(2-n)\gamma_\alpha\gamma_\mu\gamma_\beta\frac{\mu^\varepsilon}{i\pi^2}\int_0^1 dx\int_0^1 dyy\int\frac{d^nqq_\alpha q_\beta}{\left(q^2-2yq\cdot k_x\right)^3} \\
&= \int_0^1 dx\int_0^1 dy\left[(2-\varepsilon)^2\gamma_\mu\chi\left(Q^2,x\right)-(2-\varepsilon)\varepsilon\not{k}_x\gamma_\mu\not{k}_x\right] \\
&\quad\times\pi^{-\varepsilon/2}\frac{1}{2}\Gamma\left(\frac{\varepsilon}{2}\right)y^{1-\varepsilon}\frac{1}{\chi\left(Q^2,x\right)}\left[\frac{\chi\left(Q^2,x\right)}{\mu^2}\right]^{-\varepsilon/2}
\end{aligned}$$

After y-integration and Dirac algebra

$$\not{k}_x\gamma_\mu\not{k}_x=\gamma_\mu\chi\left(Q^2,x\right)-2imk_{x,\mu}$$

$$\begin{aligned}
\gamma_\alpha\gamma_\mu\gamma_\beta T_{\alpha\beta} = \gamma_\mu(1-\varepsilon)\pi^{-\varepsilon/2}\Gamma\left(\frac{\varepsilon}{2}\right)\int_0^1 dx\left[\frac{\chi\left(Q^2,x\right)}{\mu^2}\right]^{-\varepsilon/2} \\
-im\left(p_1-p_2\right)_\mu\pi^{-\varepsilon/2}\Gamma\left(1+\frac{\varepsilon}{2}\right)\int_0^1 dx\frac{1}{\chi\left(Q^2,x\right)}\left[\frac{\chi\left(Q^2,x\right)}{\mu^2}\right]^{-\varepsilon/2}
\end{aligned}$$

tensor reduces to the one-fold integrals

$$\gamma_\alpha\gamma_\mu\gamma_\beta T_{\alpha\beta} = \gamma_\mu\left(\frac{1}{\bar{\varepsilon}}-\int_0^1 dx\ln\frac{\chi\left(Q^2,x\right)}{\mu^2}-2\right)-im\left(p_1-p_2\right)_\mu F_2$$

Now use the Gordon identity

$$i\left(p_1-p_2\right)_\mu\bar{v}u=-2m\,\bar{v}\gamma_\mu u+\bar{v}\sigma_{\mu\nu}\left(p_1+p_2\right)_\nu u$$

All together

$$\Lambda_\mu=(2\pi)^4\,i\,ieQ_e\frac{e^2Q_e^2}{16\pi^2}\left[\gamma_\mu F_1+\sigma_{\mu\nu}\left(p_1+p_2\right)_\nu mF_2\right]$$

with

$$\begin{aligned}
F_1 &= -\left(Q^2+2m^2\right)\int_0^1 dx\frac{1}{\chi\left(Q^2,x\right)}\left[\frac{1}{\hat{\varepsilon}}+\ln\frac{\chi\left(Q^2,x\right)}{\mu^2}\right] \\
&\quad+\frac{1}{\bar{\varepsilon}}-\int_0^1 dx\ln\frac{\chi\left(Q^2,x\right)}{\mu^2}-2 \\
&\quad+2\left(Q^2+3m^2\right)\int_0^1 dx\frac{1}{\chi\left(Q^2,x\right)}
\end{aligned}$$

All integrals in terms of PV-functions

$$\int_0^1 dx \frac{1}{\chi(Q^2,x)}\left[\frac{1}{\hat{\varepsilon}}+\ln\frac{\chi(Q^2,x)}{\mu^2}\right] = \frac{1}{2}C_0\left(-m^2,-m^2,Q^2;m,0,m\right)$$

$$\frac{1}{\bar{\varepsilon}}-\int_0^1 dx \ln\frac{\chi(Q^2,x)}{\mu^2} = B_0\left(Q^2;m,m\right)$$

$$\left(Q^2+4m^2\right)\int_0^1 dx\frac{1}{\chi(Q^2,x)} = -2\left[B_0\left(Q^2;m,m\right)-B_0\left(-m^2;m,0\right)\right]$$

Two limiting cases:

1) $s=-Q^2\gg m^2$

$$F_1(-s;m,m) = \frac{1}{\bar{\varepsilon}}-\ln\frac{m^2}{\mu^2}-2\left(\frac{1}{\hat{\varepsilon}}+\ln\frac{m^2}{\mu^2}\right)\ln\frac{-s-i\epsilon}{m^2} - \ln^2\frac{-s-i\epsilon}{m^2}+\frac{1}{3}\pi^2+3\ln\frac{-s-i\epsilon}{m^2}$$

2) $Q^2=0$

$$F_1(0;m,m) = \frac{1}{\bar{\varepsilon}}-\frac{2}{\hat{\varepsilon}}-3\ \ln\frac{m^2}{\mu^2}+4$$

The quantity of physical interest is F_1 subtracted at zero momentum

$$F_1^{\rm sub} = F_1(-s;m,m)-F_1(0;m,m) = -2\left(\frac{1}{\hat{\varepsilon}}+\ln\frac{m^2}{\mu^2}\right)\left(\ln\frac{-s-i\epsilon}{m^2}-1\right)-\ln^2\frac{-s-i\epsilon}{m^2}+\frac{1}{3}\pi^2+3\ln\frac{-s-i\epsilon}{m^2}-4$$

The exact in all masses expression for F_1:

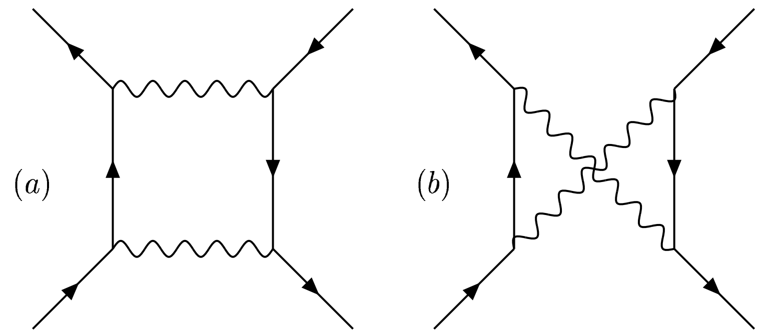
$$\frac{1}{2}F_1^{\rm sub} = \left(\frac{1}{\hat{\varepsilon}}+\ln\frac{m^2}{\mu^2}\right)\left(1+\frac{1+\beta^2}{2\beta}\ \ln\eta\right)-\frac{3}{2}\beta\ \ln\eta-2 +\frac{1+\beta^2}{\beta}\left[\mathrm{Li}_2(\eta)+\frac{1}{3}\pi^2-\frac{1}{4}\ \ln^2\eta+\ln\eta\ln(1-\eta)-\frac{i\pi}{4}\ln\frac{1-\beta^2}{4\beta^2}\right]$$

where we have introduced

$$\beta^2=1-4\frac{m^2}{s},\qquad \eta=\frac{1-\beta}{1+\beta}$$

QED box diagrams

For the annihilation $e^+e^- \to f\overline{f}$ there are two QED box diagrams: the direct (a) and the crossed (b)



The lowest order (Born) amplitude squared and summed over spins

$$\mathcal{A}_0 = \frac{1}{4}\overline{\sum_{\text{spins}}} \mid \mathcal{M}_0 \mid^2 = 2\, e^4 Q_e^2\, Q_f^2\, \frac{t^2+u^2}{s^2}$$

The corresponding contribution from the interference of the direct box diagram with the Born one

$$\mathcal{A}_{\text{int}}^{\text{dr}} = -\frac{e^6}{2\,\pi^2}\, Q_e^3 Q_f^3\, \frac{1}{s}\delta_{\gamma\gamma}^{\text{box}}\,(s,t,u)$$

where

$$\delta_{\gamma\gamma}^{\text{box}}\,(s,t,u) = u^2\mathcal{D}_{\gamma\gamma}^{+}\,(s,t,u) + t^2\mathcal{D}_{\gamma\gamma}^{-}\,(s,t,u)$$

Similarly, the crossed box is obtained with the replacement $t \leftrightarrow u$ and the change of overall sign

$$\mathcal{A}_{\text{int}}^{\text{cr}} = \frac{e^6}{2\,\pi^2}\, Q_e^3 Q_f^3\, \frac{1}{s}\delta_{\gamma\gamma}^{\text{box}}\,(s,u,t)$$

Two functions $\mathcal{D}_{\gamma\gamma}^{\pm}\,(s,t,u)$ are needed to describe boxes

$$t^2\mathcal{D}_{\gamma\gamma}^{-}\,(s,t,u) = \frac{t^2}{s}\,\left[d_0\,(s,t) + c_0\,(s;0,m_e,0) + c_0\,(s;0,m_f,0)\right]$$

$$u^2\mathcal{D}_{\gamma\gamma}^{+}\,(s,t,u) = \frac{t^2+u^2}{2\,s}\,\left[d_0\,(s,t) + c_0\,(s;0,m_e,0) + c_0\,(s;0,m_f,0)\right]$$

$$+\,(u-t)\;c_0\,(t;m_e,0,m_f) + u\left[B_0\,(-s;0,0) - B_0\,(-t;m_e,m_f)\right]$$

where

$$\begin{aligned} d_0\left(s,t\right) &= st\, D_0\left(-m_e^2,-m_e^2,-m_f^2,-m_f^2,-s,-t;0,m_e,0,m_f\right) \\ c_0\left(s;0,m_e,0\right) &= s\, C_0\left(-m_e^2,-m_e^2,-s;0,m_e,0\right) \\ c_0\left(t;m_e,0,m_f\right) &= t\, C_0\left(-m_e^2,-m_f^2,-t;m_e,0,m_f\right) \end{aligned}$$

d_0 may be split into an infrared divergent c_0 plus a finite remainder:

$$d_0\left(s,t\right) = t\,\bar{J}_{\gamma\gamma}\left(-s,-t;m_e,m_f\right) - 2\,c_0\left(t;m_e,0,m_f\right)$$

The infrared divergences in the boxes factorize into the lowest order

$$\frac{u^2}{s}\mathcal{D}^+_{\gamma\gamma}\left(s,t,u\right) + \frac{t^2}{s}\mathcal{D}^-_{\gamma\gamma}\left(s,t,u\right)\Big|_{\rm IR} = -2\,\frac{t^2+u^2}{s^2}\,c_0\left(t;m_e,0,m_f\right)$$

The ingredients for $m_e^2,\ m_f^2 \ll -t$ and $m_e^2 \ll s$:

$$\bar{J}_{\gamma\gamma}\left(-s,-t;m_e,m_f\right) = \frac{1}{t}\left[\ln\frac{m_e^2 m_f^2}{t^2}\ln\frac{-t}{s} + \frac{1}{2}\ln^2\frac{m_e^2}{-t} + \frac{1}{2}\ln^2\frac{m_f^2}{-t} + \frac{1}{3}\pi^2\right]$$

$$C_0\left(-m_e^2,-m_e^2,-s;0,m_e,0\right) = -\frac{1}{s}\left(\frac{1}{2}\,\ln^2\frac{m_e^2}{s} + \frac{1}{6}\pi^2 + i\,\pi\ln\frac{m_e^2}{s}\right)$$

$$\begin{aligned} &C_0\left(-m_e^2,-m_f^2,-t;m_e,0,m_f\right) \\ &\quad = \frac{1}{2t}\left[\ln\frac{m_e^2 m_f^2}{t^2}\left(\frac{1}{\hat{\varepsilon}} + \ln\frac{-t}{\mu^2}\right) + \frac{1}{2}\ln^2\frac{m_e^2}{-t} + \frac{1}{2}\ln^2\frac{m_f^2}{-t} + \frac{1}{3}\pi^2\right] \end{aligned}$$

$$B_0\left(-s;0,0\right) - B_0\left(-t;m_e,m_f\right) = -\ln\frac{s}{-t} + i\,\pi$$

For the total interference terms, lowest order $\times$ box diagrams we have

$$\begin{aligned} \mathcal{A}^{\rm box}_{\rm int} &= -\frac{e^6}{2\,\pi^2}\,Q_e^3 Q_f^3\, f^{\rm box}_{\gamma\gamma}\left(s,t,u\right) \\ f^{\rm box}_{\gamma\gamma}\left(s,t,u\right) &= \frac{1}{s}\left[\delta^{\rm box}_{\gamma\gamma}\left(s,t,u\right) - \delta^{\rm box}_{\gamma\gamma}\left(s,u,t\right)\right] \end{aligned}$$

$$\begin{aligned} {\rm Re}\, f^{\rm box}_{\gamma\gamma}\left(s,t,u\right) &= 2\,\frac{t^2+u^2}{s^2}\,\left(\frac{1}{\hat{\varepsilon}} + \ln\frac{s}{\mu^2}\right)\,\ln\frac{t}{u} \\ &+ \frac{t}{s}\,\ln\left(-\frac{s}{u}\right) - \frac{u}{s}\,\ln\left(-\frac{s}{t}\right) + \frac{t-u}{s}\left[\ln^2\left(-\frac{s}{t}\right) + \ln^2\left(-\frac{s}{u}\right)\right] \end{aligned}$$

Note: there are no collinear divergences and the limit of zero fermion masses can be taken.

Massless World

Two-body phase space in n-dimensions

$$\Phi_2 \;=\; (2\pi)^n\,\mu^{4-n}\int\frac{d^{n-1}p}{(2\pi)^{n-1}\,2p_0}\int\frac{d^{n-1}q}{(2\pi)^{n-1}\,2q_0}\delta^{(n)}\left(Q-p-q\right)$$

All vectors are assumed to be in n-dimensions.
Since final state particles are on-shell, $p^2=0, \quad q^2=0$, one gets:

$$\begin{aligned}\Phi_2 &= (2\pi)^{2-n}\,\mu^{4-n}\int d^np\,\delta^+(p^2)\int d^nq\,\delta^+(q^2)\delta^{(n)}\left(Q-p-q\right)\\ &= (2\pi)^{2-n}\,\mu^{4-n}\int d^np\,\delta^+(p^2)\,\delta^+((Q-p)^2)\end{aligned}$$

where $Q^2=-M^2$ and where $\delta^+(p^2)=\theta\left(p_0\right)\delta\left(p^2\right)$. Further,

$$d^np=d^{n-1}p\,dp_0\,,\qquad p^2=\sum_{i=1}^{n-1}p_i\cdot p_i-p_0^2$$

Now we go from $n-1$ rectangular coordinates to spherical coordinates involving $\mid\vec{p}\mid$ and $n-2$ angular variables

$$\begin{aligned}p_1 &= \mid\vec{p}\mid\cos\theta_1\\ p_2 &= \mid\vec{p}\mid\sin\theta_1\cos\theta_2\\ p_3 &= \mid\vec{p}\mid\sin\theta_1\sin\theta_2\cos\theta_3\\ \cdots &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ p_{n-2} &= \mid\vec{p}\mid\sin\theta_1\sin\theta_2\sin\theta_3\cdots\sin\theta_{n-3}\cos\theta_{n-2}\\ p_{n-1} &= \mid\vec{p}\mid\sin\theta_1\sin\theta_2\sin\theta_3\cdots\sin\theta_{n-3}\sin\theta_{n-2}\end{aligned}$$

with limits

$$0\le\theta_i\le\pi\quad\text{for } i=1,2,\cdots,n-3;\qquad 0\le\theta_{n-2}\le 2\pi$$

Calculating Jacobian,

$$\begin{aligned}d^{n-1}p &= \mid\vec{p}\mid^{n-2}\,d\mid\vec{p}\mid\,\sin^{n-3}\theta_1 d\theta_1\sin^{n-4}\theta_2 d\theta_2\cdots\\ &\quad\cdots\sin^2\theta_{n-4}d\theta_{n-4}\sin\theta_{n-3}d\theta_{n-3}d\theta_{n-2}\end{aligned}$$

Using

$$\int_0^\pi\sin^m\theta d\theta=\sqrt{\pi}\frac{\Gamma\left(\frac{1}{2}(m+1)\right)}{\Gamma\left(\frac{1}{2}(m+2)\right)}$$

one has

$$\Phi_2 = (2\pi)^{2-n}\,\mu^{4-n}\int \mid \vec{p}\mid^{n-3}\frac{1}{2}d\mid \vec{p}\mid^2\,dp_0\pi^{n/2-2}\frac{\Gamma\left(\frac{1}{2}(n-3)\right)}{\Gamma\left(\frac{1}{2}(n-2)\right)}\frac{\Gamma\left(\frac{1}{2}(n-4)\right)}{\Gamma\left(\frac{1}{2}(n-3)\right)}$$
$$\cdots\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(2\right)}\frac{\Gamma\left(1\right)}{\Gamma\left(\frac{3}{2}\right)}2\pi\delta^+(\mid \vec{p}\mid^2 - p_0^2)\,\delta^+(-M^2+2Mp_0)\sin^{n-3}\theta_1 d\theta_1$$

with

$$\mid \vec{p}\mid^2 = p_1^2+p_2^2+\cdots+p_{n-1}^2$$

and simplifying we arrive at important intermediate result

$$\Phi_2 = (2\pi)^{4-n}\frac{\pi^{n/2-2}}{8\pi}\frac{\mid \vec{p}\mid}{M}\left(\frac{\mid \vec{p}\mid}{\mu}\right)^{n-4}\frac{1}{\Gamma\left(\frac{1}{2}n-1\right)}\int_0^\pi \sin^{n-4}\theta_1 d\cos\theta_1$$

For infrared regularization $n=4+\varepsilon'$, and $\cos\theta_1 = y$, $\mid \vec{p}\mid = p_0 = M/2$

$$\Phi_2 = (2\pi)^{-\varepsilon'}\frac{\pi^{\varepsilon'/2}}{16\pi}\left(\frac{M}{2\mu}\right)^{\varepsilon'}\frac{1}{\Gamma\left(1+\varepsilon'/2\right)}\int_{-1}^{1}\left(1-y^2\right)^{\varepsilon'/2}dy\,,$$

Further, with $z=\frac{1+y}{2}$

$$\int_{-1}^{1}\left(1-y^2\right)^{\varepsilon'/2}dy = 2^{1+\varepsilon'}\int_0^1\left[z\left(1-z\right)\right]^{\varepsilon'/2}dz$$
$$= 2^{1+\varepsilon'}\mathrm{B}\left(1+\frac{1}{2}\varepsilon',1+\frac{1}{2}\varepsilon'\right) = 2^{1+\varepsilon'}\frac{\Gamma\left(1+\varepsilon'/2\right)^2}{\Gamma\left(2+\varepsilon'\right)}$$

finally get a presentation convenient for expansions in ε'

$$\Phi_2 = \frac{1}{8\pi}\left(\frac{M^2}{\mu^2}\right)^{\varepsilon'/2}\frac{(2\pi)^{-\varepsilon'}\,\pi^{\varepsilon'/2}\Gamma\left(1+\varepsilon'/2\right)}{\left(1+\varepsilon'\right)\Gamma\left(1+\varepsilon'\right)}$$

For fun of it, using the so-called *duplication Legendre formula*:

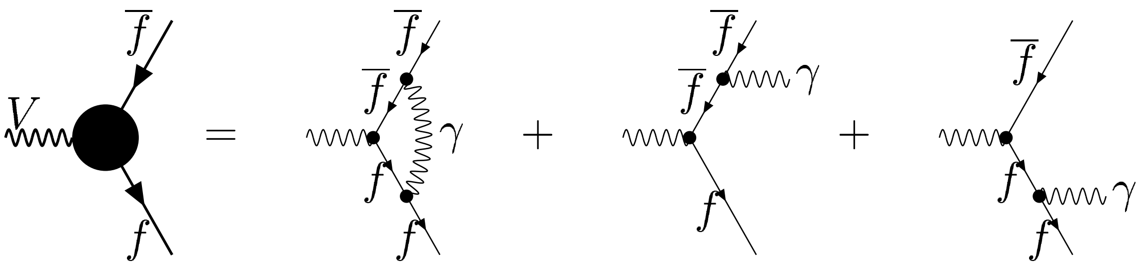
$$\sqrt{\pi}\Gamma\left(2z\right) = 2^{2z-1}\Gamma\left(z\right)\Gamma\left(z+\frac{1}{2}\right)$$

it can be reduced further on

$$\Phi_2 = \frac{1}{16\pi}\left(\frac{M^2}{\mu^2}\right)^{\varepsilon'/2}\frac{2^{-2\varepsilon'}\pi^{1/2-\varepsilon'/2}}{\Gamma\left(3/2+\varepsilon'/2\right)}$$

that is not convenient for expansions, however.

Calculation of Z decay width with QED radiative corrections



QED vertex

For on-shell massless fermions and on-shell vector boson:

$p_1^2 = p_2^2 = 0, \quad \bar{v}(p_2)\, \not{p}_2 = 0, \quad \not{p}_1\, u(p_1) = 0, \quad Q^2 = 2p_1 \cdot p_2 = -M_V^2$

$$\Lambda_\mu = i\,(eQ_e)^3\,\mu^{4-n} \int d^n q \frac{1}{q^2\,(q+p_1)^2\,(q-p_2)^2} N_\mu$$

$$N_\mu = -4p_1\cdot p_2\gamma_\mu + 2\,(\not{p}_1\gamma_\alpha\gamma_\mu - \gamma_\mu\gamma_\alpha\not{p}_2)\,q_\alpha + (2-n)\,\gamma_\alpha\gamma_\mu\gamma_\beta q_\alpha q_\beta$$

In massless case: $k_x = xp_2 - (1-x)\,p_1\,, \quad \chi\left(Q^2, x\right) = Q^2 x\,(1-x)$

$$\Lambda_\mu = i\,(eQ_e)^3\left[-Q^2 S\gamma_\mu + 2\,(\not{p}_1\gamma_\alpha\gamma_\mu - \gamma_\mu\gamma_\alpha\not{p}_2)\,V_\alpha + \gamma_\alpha\gamma_\mu\gamma_\beta T_{\alpha\beta}\right]$$

For the Scalar we now have to continue integration in n dimensions

$$\begin{aligned} -Q^2 S &= -2\frac{\pi^{\varepsilon'/2}}{\varepsilon'}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon'/2}\int_0^1 dx x^{\varepsilon'/2-1}\,(1-x)^{\varepsilon'/2-1} \\ &= -2\frac{\pi^{\varepsilon'/2}}{\varepsilon'}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon'/2}\mathrm{B}\left(\frac{\varepsilon'}{2},\frac{\varepsilon'}{2}\right) \\ &= -2\frac{\pi^{\varepsilon'/2}}{\varepsilon'}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon'/2}\frac{\Gamma^2\left(\varepsilon'/2\right)}{\Gamma\left(\varepsilon'\right)} \end{aligned}$$

Similarly for Vector (we use *infrared regulator* ε')

$$2\,(\not{p}_1\gamma_\alpha\gamma_\mu - \gamma_\mu\gamma_\alpha\not{p}_2)\,V_\alpha = \gamma_\mu\,2\frac{\pi^{\varepsilon'/2}}{1+\varepsilon'}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon'/2}\frac{\Gamma^2\left(\varepsilon'/2\right)}{\Gamma\left(\varepsilon'\right)}$$

Massless Vector is **not** finite and we may not set $\varepsilon' = 0$.

Mass singularities, collinear divergences.

Tensor

$$\gamma_\alpha \gamma_\mu \gamma_\beta T_{\alpha\beta} = \gamma_\mu \left(1+\varepsilon'\right) \pi^{\varepsilon'/2} \Gamma\left(-\frac{\varepsilon'}{2}\right) \left(\frac{Q^2}{\mu^2}\right)^{\varepsilon'/2} \int_0^1 dx x^{\varepsilon'/2} \left(1-x\right)^{\varepsilon'/2}$$

Expansions should be performed up to ε'^2

$$\begin{aligned} \Gamma\left(1+x\right) &= 1 - \gamma\, x + \frac{1}{2}\left[\zeta(2)+\gamma^2\right] x^2 + \mathcal{O}\left(x^3\right), \qquad \zeta(2) = \frac{\pi^2}{6} \\ a^x &= 1 + \left(\ln a\right) x + \frac{1}{2}\left(\ln a\right)^2 x^2 + \mathcal{O}\left(x^3\right) \end{aligned}$$

Notations

$$\begin{aligned} \frac{1}{\hat{\varepsilon}} &= \frac{2}{\varepsilon'} + \gamma + \ln \pi\,, \qquad \overline{\gamma} = \gamma + \ln \pi \\ z_V &= \ln \frac{-M_V^2 - i\,\epsilon}{\mu^2} = \ln \frac{M_V^2}{\mu^2} - i\,\pi \end{aligned}$$

In massless case only F_1 remains

$$\Lambda_\mu = \left(2\pi\right)^4 i\, ieQ_e \frac{e^2}{16\pi^2} \gamma_\mu F_1$$

F_1 and its ingredients

$$\begin{aligned} \text{Scalar} &= -\frac{2}{\hat{\varepsilon}^2} + \frac{2}{\hat{\varepsilon}}\left(\overline{\gamma} - z_V\right) - \overline{\gamma} - z_V^2 + \zeta(2) \\ \text{Vector} &= \frac{4}{\hat{\varepsilon}} - 8 + 4 z_V \\ \text{Tensor} &= -\frac{1}{\hat{\varepsilon}} - z_V \\ F_1 &= -2\frac{1}{\hat{\varepsilon}^2} + \frac{2}{\hat{\varepsilon}}\left(\overline{\gamma} - z_V + \frac{3}{2}\right) - \overline{\gamma}^2 - z_V^2 + \zeta(2) + 3 z_V - 8 \end{aligned}$$

Note:

– F_1 at zero momentum is zero, $\left(\frac{Q^2=0}{\mu^2}\right)^{\varepsilon'/2} = 0$, for $\varepsilon' > 0$, infrared regularization;

– in tensor integral we face a migration of ultraviolet pole into an infrared pole;

– physical origin of double poles: infrared$\times$mass.

Fermionic self-energy in massless world

Massive expression reduces to

$$\Sigma(\not p) = -e^2 Q_e^2 \int d^n q \frac{\gamma_\mu i \not q \gamma_\mu}{(q^2 - i\epsilon)\left[(q+p)^2 - i\epsilon\right]}$$

$$= i\pi^2 \left(-e^2 Q_e^2\right)(2-n)\,\pi^{n/2-2}\Gamma\left(\frac{n}{2}-2\right)\int_0^1 dx x \left[\frac{p^2 x(1-x)}{\mu^2}\right]^{\varepsilon'/2} i\not p$$

$$= i\pi^2 \left(e^2 Q_e^2\right)\pi^{\varepsilon'/2}\,(2+\varepsilon')\,\Gamma\left(-\frac{\varepsilon'}{2}\right)\,\mathrm{B}\left(2+\frac{\varepsilon'}{2}, 1+\frac{\varepsilon'}{2}\right)\left(\frac{p^2}{\mu^2}\right)^{\varepsilon'/2} i\not p$$

Important:
fermionic self-energy in massless world vanishes on fermionic mass-shell, i.e. at $p^2 = 0$ (for the same reason as $F_1 = 0$)

Virtual correction in n-dimensions

$$\text{Virtual} = \frac{1}{n-1}\frac{1}{2M_V}\overline{\sum_{\text{spins}}} 2\,\mathrm{Re}\left(A^{\text{Born}} A^{1L}\right)$$

the factor $1/(n-1)$ follows from averaging over the V polarizations. For a correct treatment of the factors 2π, we have not to forget

$$d^n q \to \frac{(2\pi)^4}{(2\pi)^n} d^n q = (2\pi)^{-\varepsilon'} d^n q$$

Further

$$A^{1L} = \frac{e^2}{16\pi^2} A^{\text{Born}} F_1$$

$$\text{Virtual} = |A^{\text{Born}}|^2 \frac{\alpha}{\pi}\delta^V$$

$$\overline{\sum_{\text{spins}}} |A^{\text{Born}}|^2 \propto \left(1 + \frac{\varepsilon'}{2}\right)$$

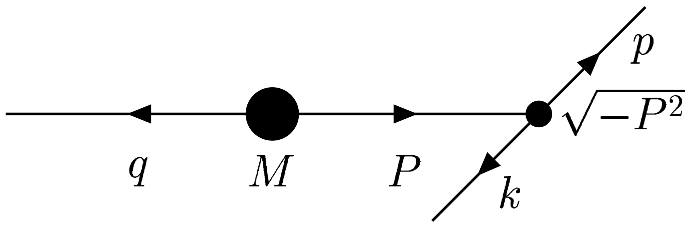
After expanding of all ingredients:

$$\delta^V = -\frac{1}{\hat\varepsilon^2} - \frac{2}{\hat\varepsilon}\left(L_V - \frac{19}{12}\right) - 2L_V^2 - 2\overline{\gamma} L_V + 5\zeta(2) - \overline{\gamma}^2 + \frac{19}{3}L_V + \frac{19}{6}\overline{\gamma} - \frac{173}{18}, \qquad \text{with} \quad L_V = \ln\frac{M_V^2}{(2\pi\mu)^2}$$

Three-body phase space in n-dimensions

$$
\begin{aligned}
d\Phi_3 &= (2\pi)^n \mu^{8-2n} \frac{d^{n-1}p}{(2\pi)^{n-1}2p_0}\frac{d^{n-1}q}{(2\pi)^{n-1}2q_0}\frac{d^{n-1}k}{(2\pi)^{n-1}2k_0} \\
&\quad \times \delta^{(n)}(Q-p-q-k) \\
&= M_V^2 (2\pi)^{3-2n}\mu^{8-2n} d^n p\, \delta^+(p^2) d^n q\, \delta^+(q^2) d^n k\, \delta^+(k^2) \\
&\quad \times \delta^{(n)}(Q-p-q-k)\, d^n P \delta^{(n)}(P-p-k) \\
&\quad \times d\left(-P^2\right)\delta^+\left(-P^2+(p+k)^2\right)
\end{aligned}
$$

Kinematical cascade



$$
\begin{aligned}
d\Phi_3 &= \frac{1}{2\pi} d\left(-P^2\right) \\
&\times\ (2\pi)^{2-n}\mu^{4-n} d^n q\, \delta^+(q^2) d^n P\, \delta^+\left(-P^2+(Q-q)^2\right)\delta^{(n)}(Q-P-q) \\
&\times\ (2\pi)^{2-n}\mu^{4-n} d^n p\, \delta^+(p^2) d^n k\, \delta^+(k^2)\delta^{(n)}(P-p-k) \\
&= \frac{1}{2\pi} d\left(-P^2\right) \\
&\times\ (2\pi)^{-\varepsilon'} \frac{\pi^{\varepsilon'/2}}{8\pi}\left(\frac{M^2+P^2}{2M^2}\right)\left(\frac{M^2+P^2}{2M\mu}\right)^{\varepsilon'} \frac{1}{\Gamma(1+\varepsilon'/2)} \sin^{\varepsilon'}\theta\, d\cos\theta \\
&\times\ (2\pi)^{-\varepsilon'} \frac{\pi^{\varepsilon'/2}}{16\pi}\left(\frac{\sqrt{-P^2}}{2\mu}\right)^{\varepsilon'} \frac{1}{\Gamma(1+\varepsilon'/2)} \sin^{\varepsilon'}\theta_1\, d\cos\theta_1
\end{aligned}
$$

Reminder

$$
\Phi_2 = (2\pi)^{4-n}\frac{\pi^{n/2-2}}{8\pi}\frac{|\vec{p}\,|}{M}\left(\frac{|\vec{p}\,|}{\mu}\right)^{n-4}\frac{1}{\Gamma\left(\frac{1}{2}n-1\right)}\int_0^\pi \sin^{n-4}\theta_1\, d\cos\theta_1
$$

Two angular integrations should be treated differently.
First might be taken (matrix element squared is independent of it)

$$\int_0^\pi \sin^{\varepsilon'} \theta d\cos\theta = \int_{-1}^1 \left(1-y^2\right)^{\varepsilon'/2} dy = 2^{1+\varepsilon'} \frac{\Gamma\left(1+\varepsilon'/2\right)^2}{\Gamma\left(2+\varepsilon'\right)}$$

Second, with $z = \dfrac{1+y}{2}$

$$\int_0^\pi \sin\theta_1^{\varepsilon'} d\cos\theta_1 = \int_{-1}^1 \left(1-y^2\right)^{\varepsilon'/2} dy = 2^{1+\varepsilon'} \left[z\left(1-z\right)\right]^{\varepsilon'/2} dz$$

should be kept untaken.
Substituting angular integrals,

$$\begin{aligned} d\Phi_3 &= \frac{1}{2^7\pi^3} \frac{\left(2\pi\right)^{-2\varepsilon'} \pi^{\varepsilon'}}{\Gamma\left(2+\varepsilon'\right)} \\ &\times\ d\left(-P^2\right) \frac{M^2+P^2}{M^2} \left(\frac{M^2+P^2}{M\mu}\right)^{\varepsilon'} \left(\frac{\sqrt{-P^2}}{\mu}\right)^{\varepsilon'} \left[z\left(1-z\right)\right]^{\varepsilon'/2} dz \end{aligned}$$

Introducing $-P^2 = xM^2$, we finally get

$$\Phi_3 = \frac{M^2}{2^7\pi^3} \left(\frac{M^2}{\mu^2}\right)^{\varepsilon'} \frac{\left(2\pi\right)^{-2\varepsilon'} \pi^{\varepsilon'}}{\Gamma\left(2+\varepsilon'\right)} \int_0^1 dx x^{\varepsilon'/2} \left(1-x\right)^{1+\varepsilon'} \int_0^1 dz \left[z\left(1-z\right)\right]^{\varepsilon'/2}$$

The radiative decay $V \to f\overline{f}\gamma$

The radiative process $V(Q) \to f(p) + \overline{f}(q) + \gamma(k)$.

<u>The kinematics</u> may be specified in terms of two invariants, x and y:

$$xM_V^2 = -(p+k)^2, \qquad (y+1)M_V^2 = -(Q+k)^2$$

Indeed

$$\begin{aligned} -2\,p\cdot k &= xM_V^2, & -2\,q\cdot k &= (y-x)M_V^2 \\ -2\,Q\cdot k &= yM_V^2, & -2\,p\cdot q &= (1-y)M_V^2 \\ -2\,Q\cdot q &= (1-x)M_V^2, & -2\,Q\cdot p &= (1-y+x)M_V^2 \end{aligned}$$

The bremsstahlung amplitude

$$\mathcal{M}^{\rm brem} = -i\,e^2\,\bar{u}\,(p)\left[\not{e}\frac{(\not{q}+\not{k})}{(q+k)^2}\not{\epsilon} - \not{\epsilon}\frac{(\not{p}+\not{k})}{(p+k)^2}\not{e}\right]v\,(q)$$

where $e(Q)$ and $\epsilon(k)$ are the V and photon polarization vectors.

The amplitude squared can be expressed in terms of invariants x, z:

$$\overline{\sum_{\rm spins}}|\,\mathcal{M}^{\rm brem}|^2 = e^4\,\varepsilon^*\left\{2\left(\frac{1}{zx}-\frac{1}{x}-1\right)+\frac{\varepsilon^*}{8}\left[\frac{1}{z}\frac{x}{1-x}+2+z\left(\frac{1}{x}-1\right)\right]\right\}$$

Here, $\varepsilon^* = 8 + 4\,\varepsilon'$ and

$$y = (1-x)\,z + x$$

The 3-body phase-space integral in n dimensions,

$$\begin{aligned} \int d\Phi_3 &= \int_0^1 dx \int_0^1 dz\,\Phi_3 \\ \Phi_3 &= \frac{M_V^2}{2^6\pi^3}\left(\frac{M_V^2}{\mu^2}\right)^{\varepsilon'}\frac{(2\pi)^{-2\varepsilon'}\,\pi^{\varepsilon'}}{(1+\varepsilon')\,\Gamma\,(1+\varepsilon')}\,(1-x)^{1+\varepsilon'}\,[z\,(1-z)]^{\varepsilon'/2} \end{aligned}$$

x and z integrations are left undone, because of the explicit dependence of the integrand. One should also include an extra factor:

$$\frac{1}{n-1}\frac{1}{2\,M_V}$$

from averaging over V boson spin.

The complete bremsstrahlung contribution is the product of the amplitude squared × the phase-space factor integrated over the $x,\ z$. All bremsstrahlung integrals can be easily performed in n dimensions and at the very end one expands around $\varepsilon' = 0$.

$$\int_0^1 dx \int_0^1 dz\, x^{\varepsilon'/2} (1-x)^{1+\varepsilon'} \left[z(1-z)\right]^{\varepsilon'/2} \mathcal{A}^{\text{brem}} =$$
$$= e^4 \left[\left(\frac{8}{\varepsilon'}\right)^2 - \frac{16}{\varepsilon'} + 52 - 48\zeta(2) \right]$$

If one include phase space and all relevant factors:

$$\begin{aligned}\delta^R \;=\; & \frac{1}{\hat{\varepsilon}^2} + \frac{2}{\hat{\varepsilon}} \left(L_V - \frac{19}{12} \right) + 2L_V^2 + 2\overline{\gamma} L_V - 5\zeta(2) + \overline{\gamma}^2 \\ & - \frac{19}{3} L_V - \frac{19}{6} \overline{\gamma} + \frac{373}{36}\end{aligned}$$

The complete expression is the sum of virtual and real contributions.

$$\Gamma^{\text{QED}} = \Gamma^{\text{Born}} \left(1 + \frac{\alpha}{\pi} \delta^{\text{QED}} \right)$$

with

$$\delta^{\text{QED}} \;=\; \delta^R + \delta^V = \frac{373}{36} - \frac{173}{18} = \frac{3}{4}$$

Some conclusions:

- All the poles (infrared and mass singularities) and the logarithms cancel in the combined expression;
- KLN theorem for inclusive setup;
- Renormalization was not needed in this example.

Summary of Level 3

1) Standard Model, its Fields and Lagrangian

Feynman Rules $\rightarrow$ *building* of diagrams

2) Regularization, N-point functions

A, B, C, D-functions $\rightarrow$ *calculation* of diagrams

3) QED diagrams, *building blocks:*

– photonic and fermionic self energies
– vertex and boxes

4) First feeling of renormalization - subrtaction at 0 momentum

5) Example of calculation of RC's for the decay $V \rightarrow f\overline{f}$ in massless QED

– well-known correction $\dfrac{3\alpha}{4\pi}$
– first feeling of divergency cancellation
– Why renormalization? Not clear yet...

Bosonic self-energies and transitions

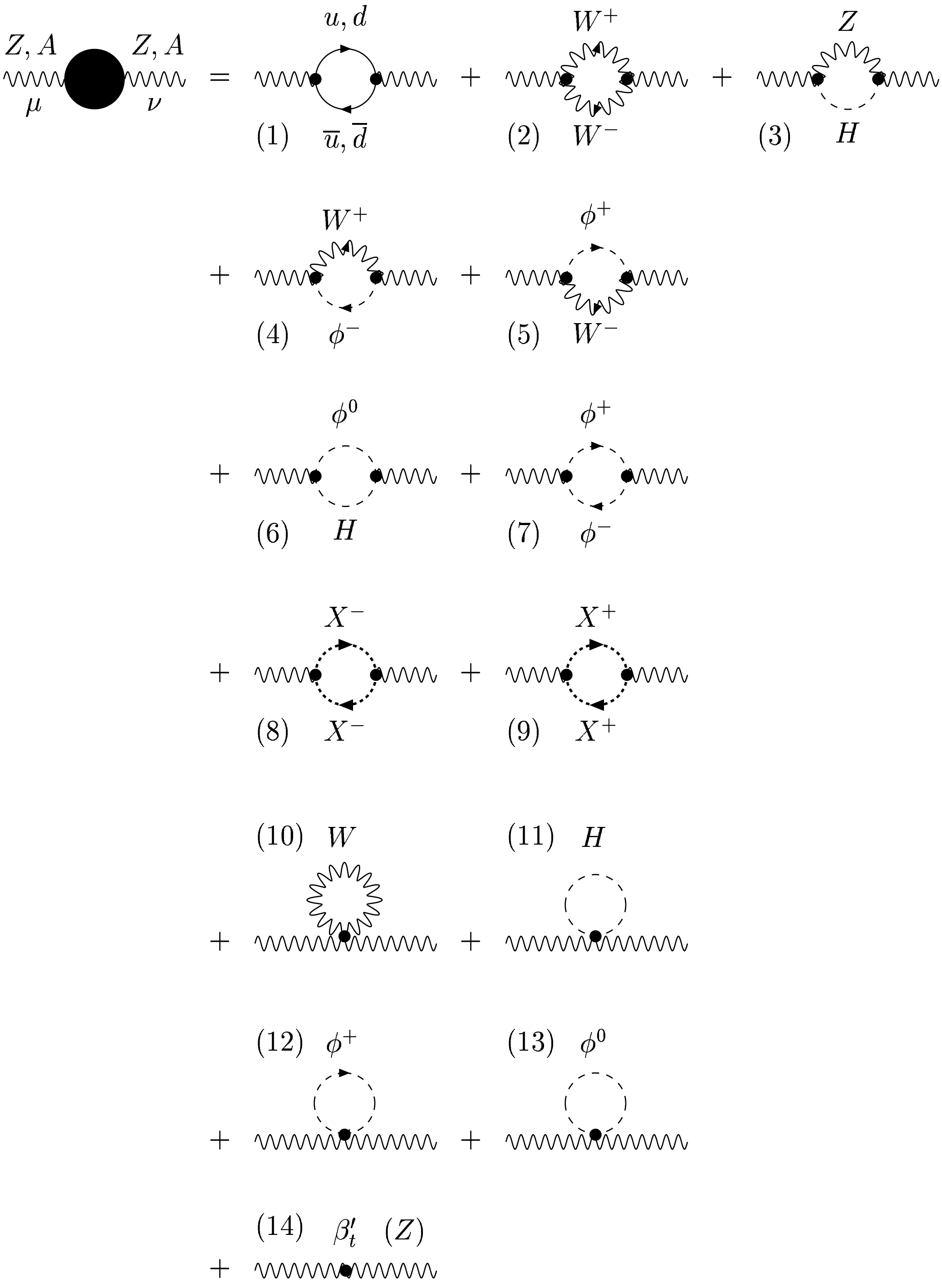


Figure 1: (Z, A)-boson self-energy; $Z - A$ transition

Let $S_{WW}\,(p^2)$ be the $\delta_{\mu\nu}$ part of the *sum* of diagrams:

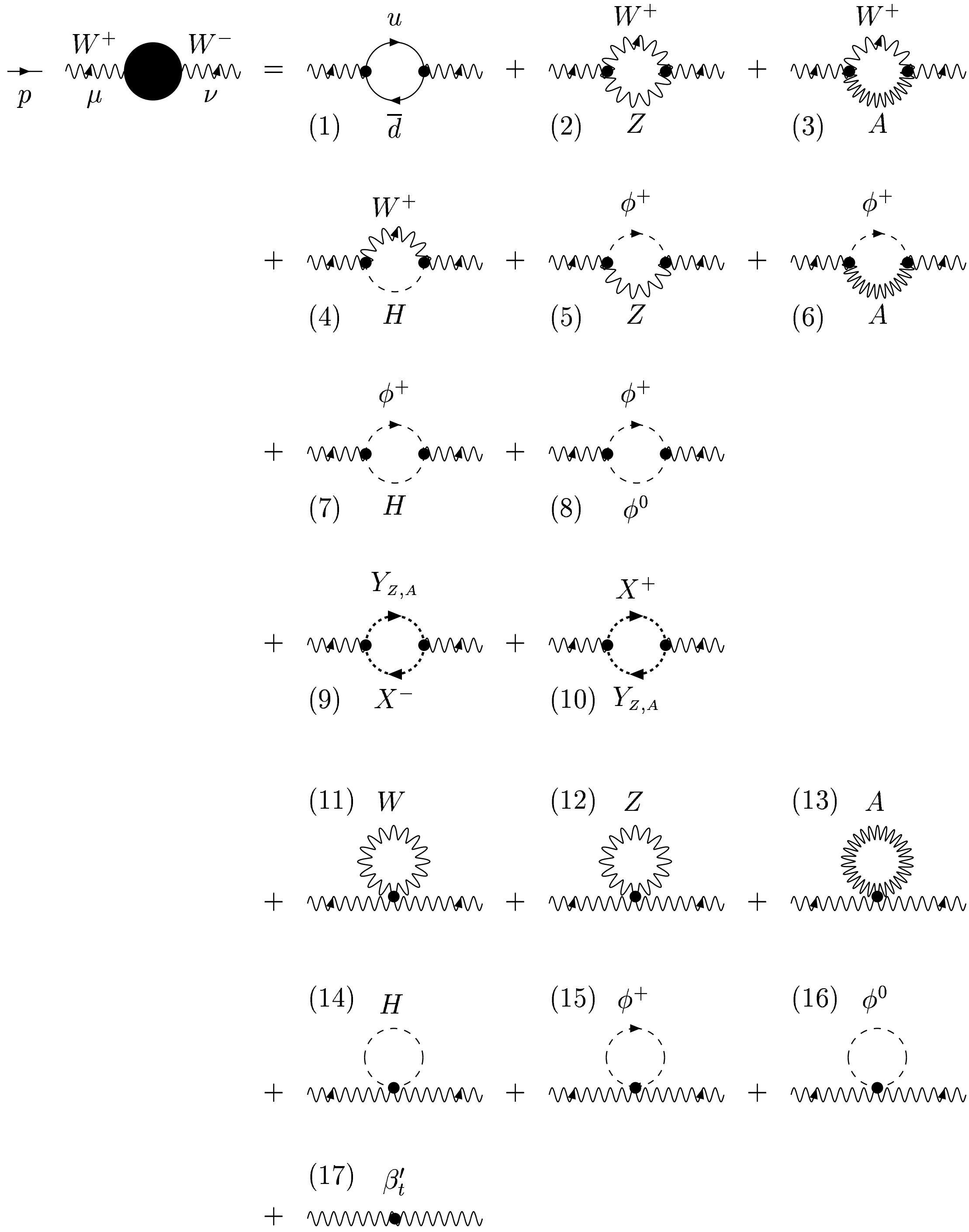


Figure 2: W-boson self-energy

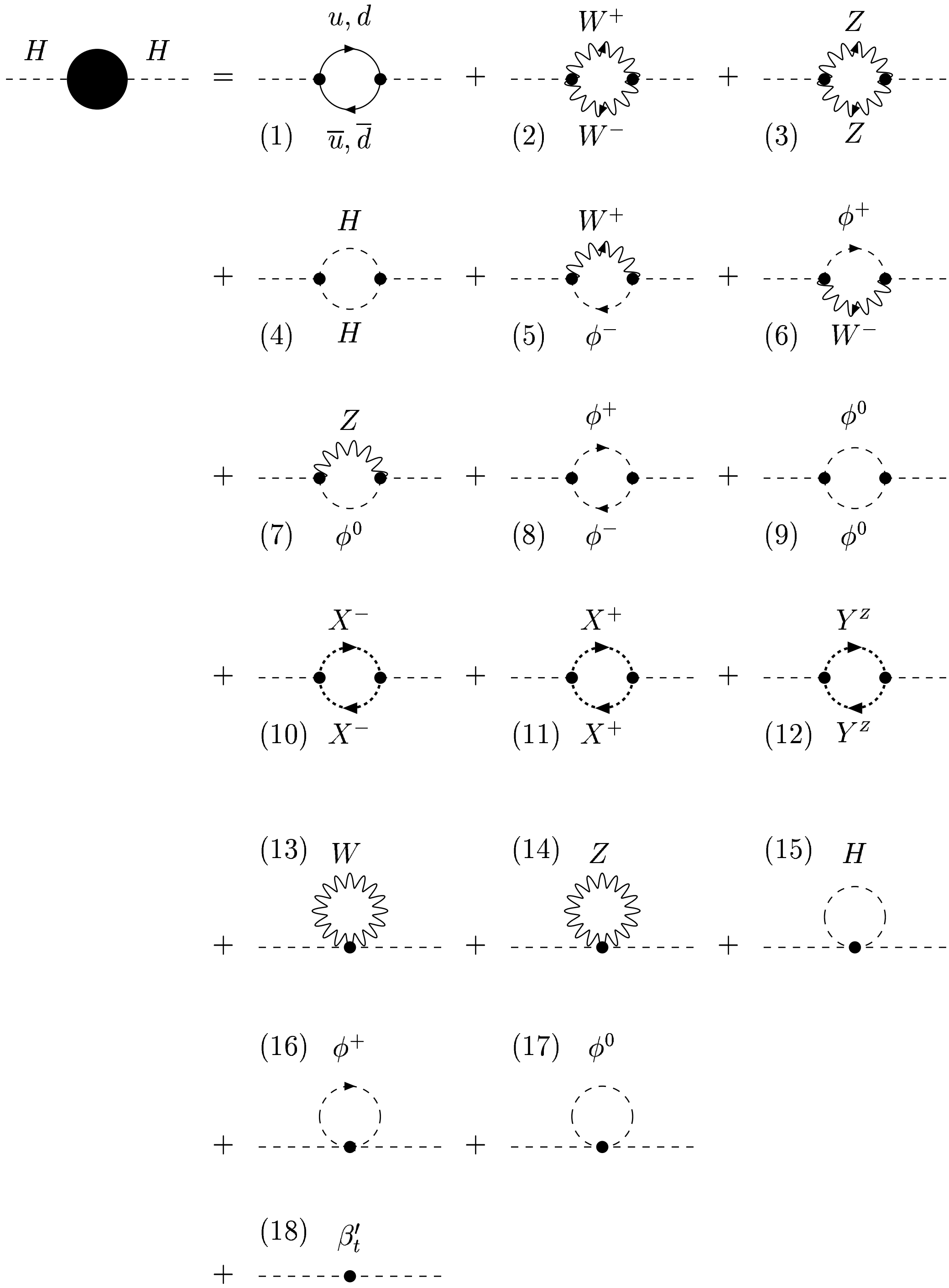


Figure 3: H-boson self-energy

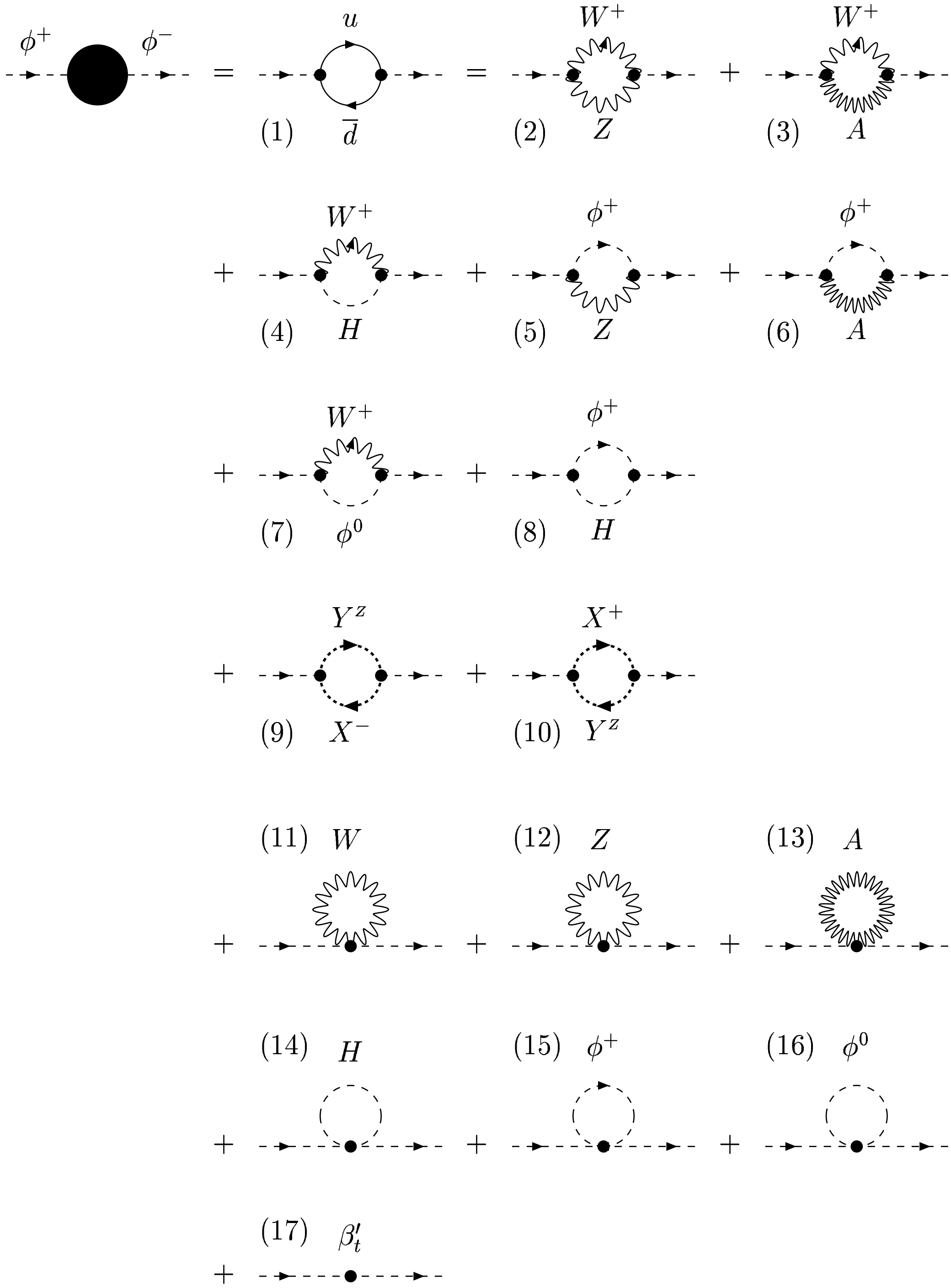


Figure 4: $\phi^{\pm}$-boson self-energy

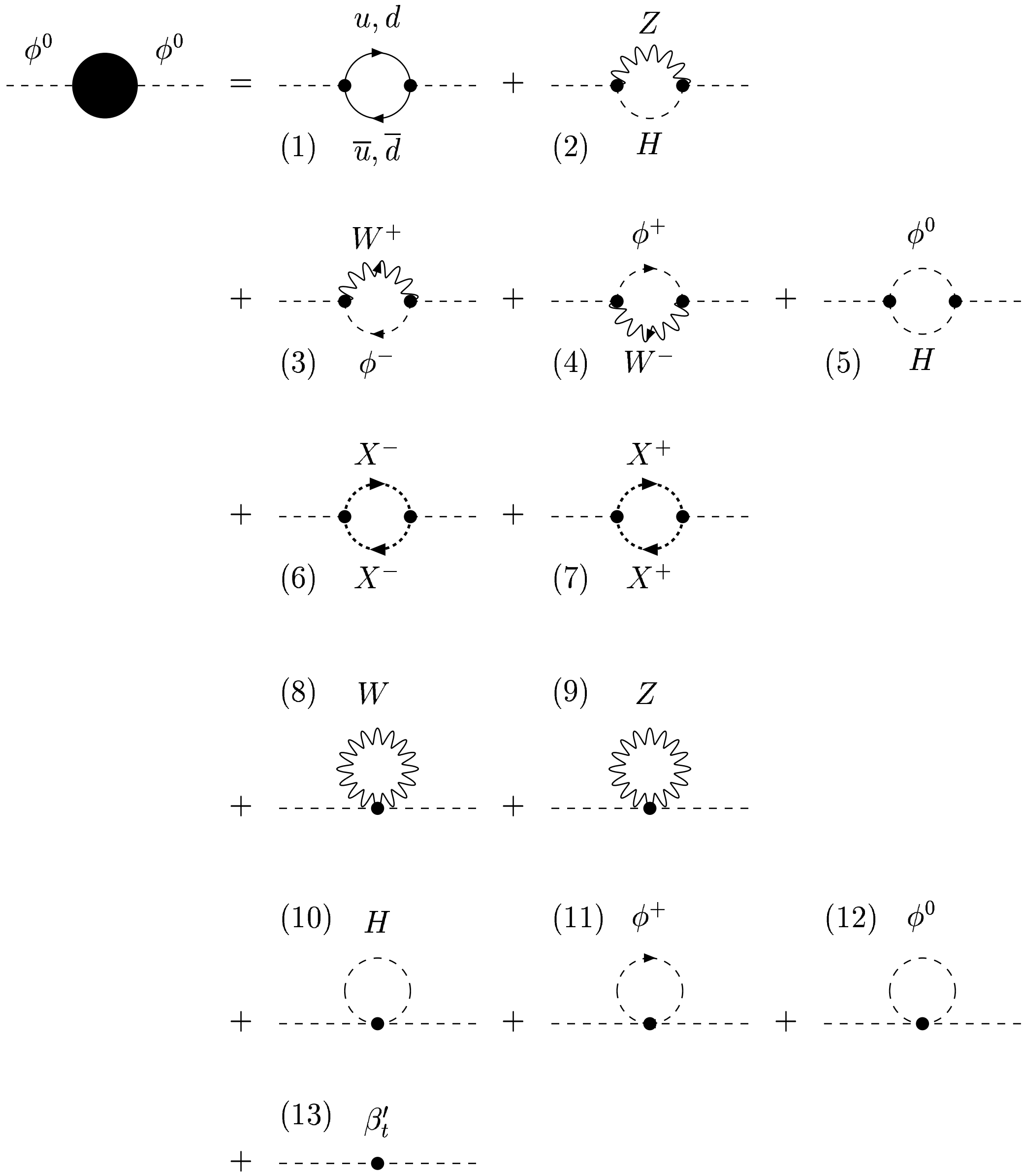


Figure 5: ϕ^0-boson self-energy

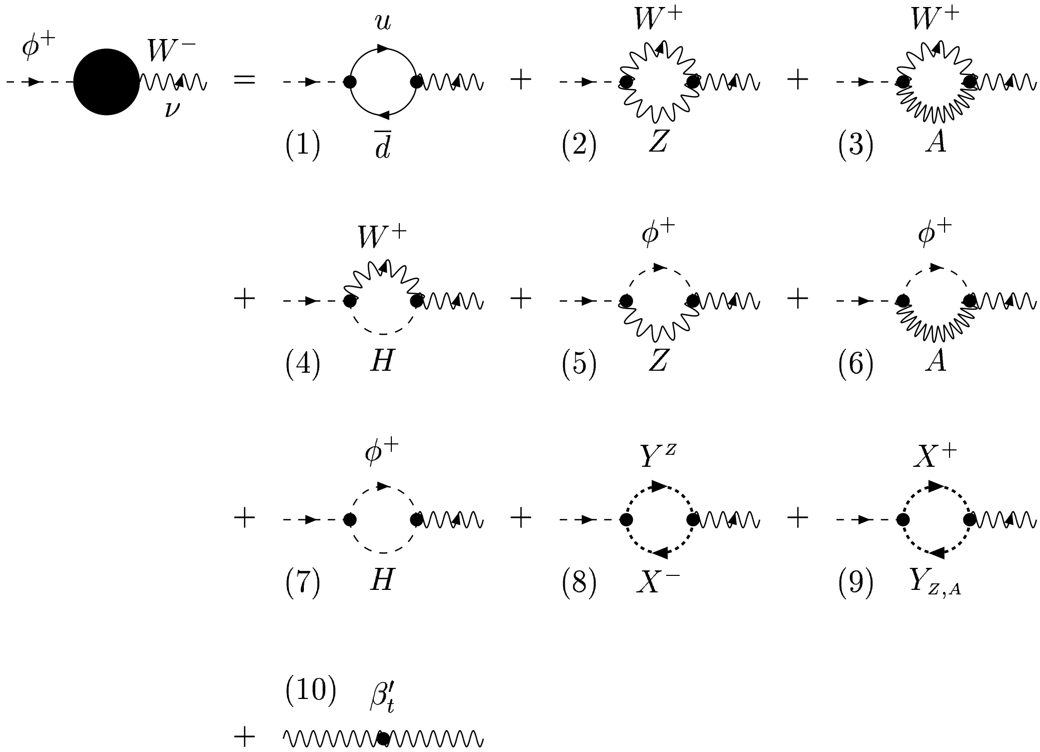


Figure 6: $\phi - W$ transition

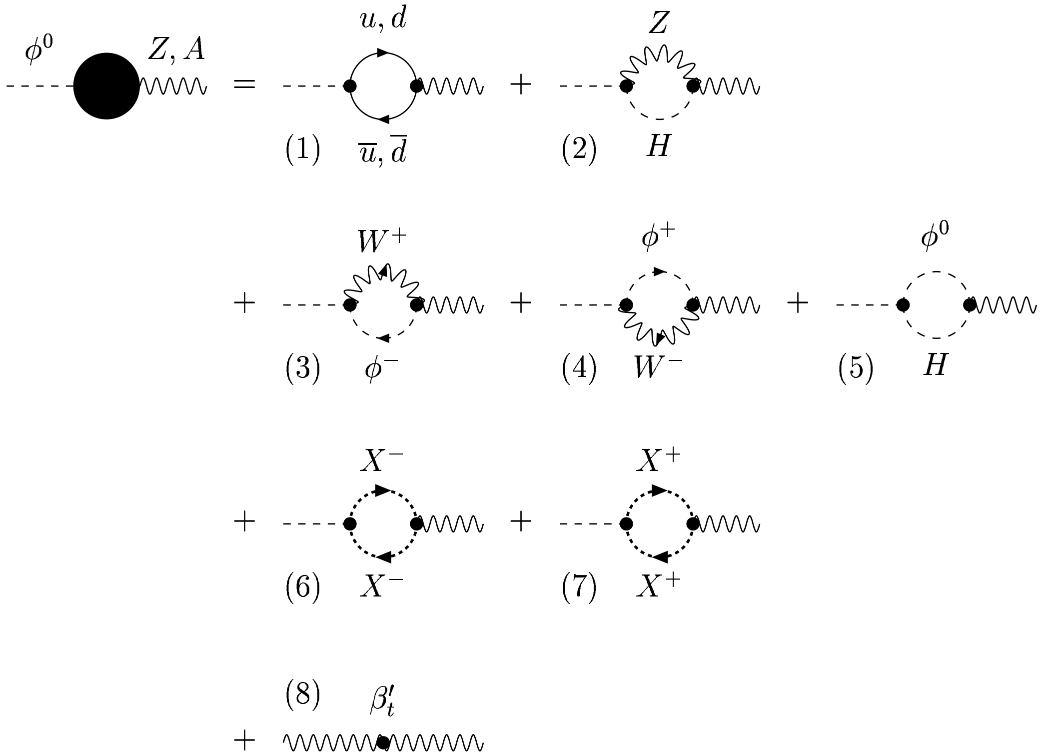


Figure 7: $\phi^0 - (Z, A)$ transition

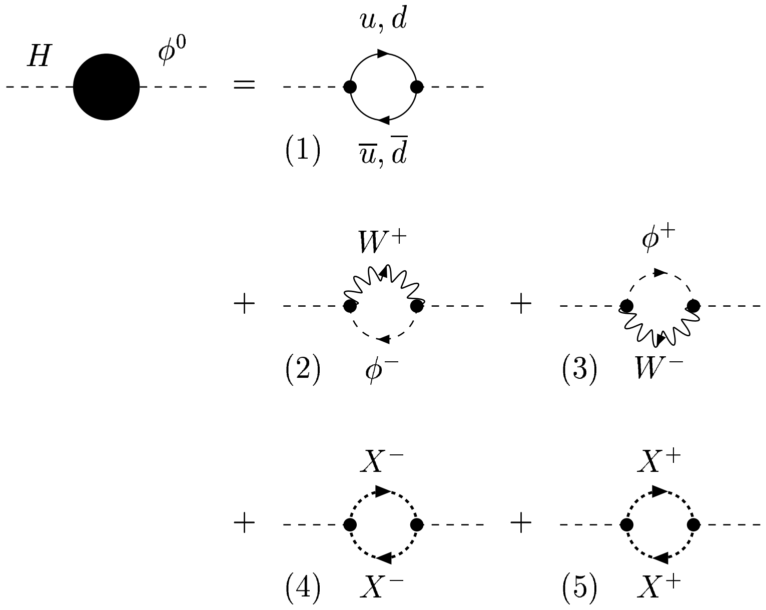


Figure 8: $H - \phi^0$ transition

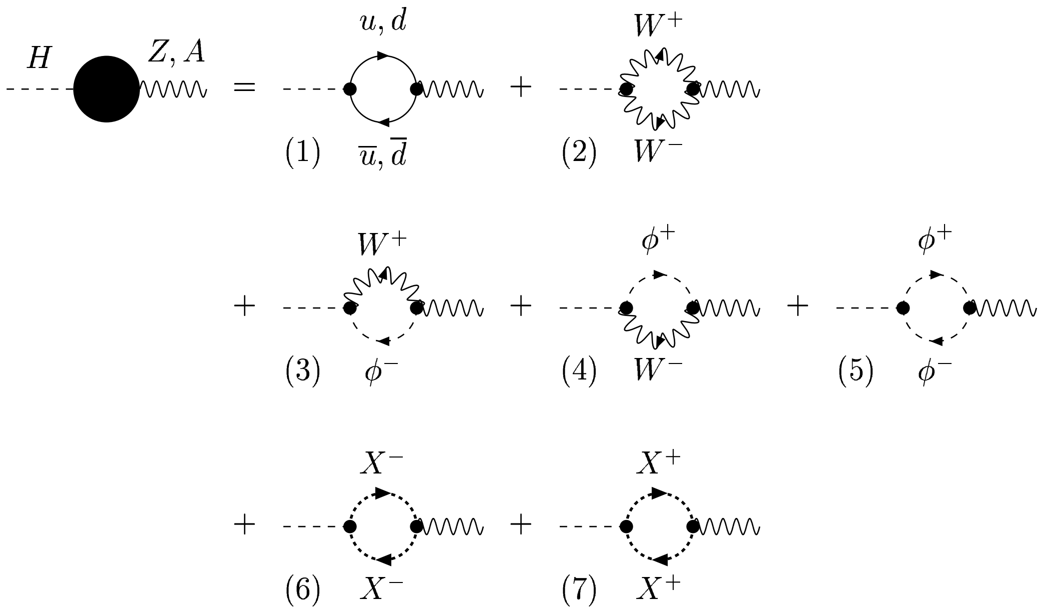


Figure 9: $H - (Z, A)$ transition

$\underline{R_\xi \text{ gauge.}}$ $\delta_{\mu\nu}$ part of the W-boson self-energy:

$$S_{WW}(p^2) = \frac{g^2}{16\pi^2}\Sigma^{\xi}_{WW}(p^2), \qquad \Sigma^{\xi}_{AB}(p^2) = \Sigma^{(1)}_{AB}(p^2) + \Sigma^{\mathrm{add}}_{AB}(p^2)$$

$$\begin{aligned}
\Sigma^{(1)}_{WW} \;=\; & \frac{M^2}{12}\Bigg\{ -\left[\frac{s_\theta^4}{c_\theta^4}\left(1+8c_\theta^2\right)\frac{M^2}{p^2} - \frac{10}{c_\theta^2} + 54 + 16c_\theta^2 + \left(1-40c_\theta^2\right)\frac{p^2}{M^2}\right] \\
& \qquad\qquad \times B_0\left(p^2; M, M_0\right) \\
& - \left[\left(1-\frac{M_H^2}{M^2}\right)^2\frac{M^2}{p^2} - 10 + 2\frac{M_H^2}{M^2} + \frac{p^2}{M^2}\right] B_0\left(p^2; M_H, M\right) \\
& -8s_\theta^2\left(\frac{M^2}{p^2} + 2 - 5\frac{p^2}{M^2}\right) B_0\left(p^2; 0, M\right) \\
& + \left[\left(\frac{1}{c_\theta^2} - 2 + \frac{M_H^2}{M^2}\right)\frac{M^2}{p^2} - 14 + 36\frac{M^2}{M_H^2}\right]\frac{1}{M^2}A_0\left(M\right) \\
& - \left[\frac{s_\theta^2}{c_\theta^2}\left(1+8c_\theta^2\right)\frac{M^2}{p^2} - 1 - 18\frac{M_0^2}{M_H^2} + 16c_\theta^2\right]\frac{1}{M^2}A_0\left(M_0\right) \\
& + \left(\frac{M^2 - M_H^2}{p^2} + 7\right)\frac{1}{M^2}A_0\left(M_H\right) \\
& +12\left(\frac{1}{c_\theta^4} + 2\right)\frac{M^2}{M_H^2} - 2\left(\frac{1}{c_\theta^2} + 18 + \frac{M_H^2}{M^2} - \frac{2}{3}\frac{p^2}{M^2}\right)\Bigg\}
\end{aligned}$$

$$\begin{aligned}
\Sigma^{\mathrm{add}}_{WW} \;=\; & \frac{M^2+p^2}{12}\Bigg\{\left[s_\theta^4\frac{M^2}{p^2} - 1 + 4c_\theta^2 + c_\theta^4 - c_\theta^2\left(2+c_\theta^2\right)\frac{p^2}{M^2} - c_\theta^4\frac{p^4}{M^4}\right] \\
& \qquad\qquad \times\left[B_0\left(p^2; \xi M, \xi_Z M_0\right) - B_0\left(p^2; M_0, \xi M\right)\right] \\
& +2\left(\frac{s_\theta^4}{c_\theta^2}\frac{M^2}{p^2} - 10 + 8c_\theta^2 - 5c_\theta^2\frac{p^2}{M^2}\right) \\
& \qquad\qquad \times\left[B_0\left(p^2; M_0, \xi M\right) - B_0\left(p^2; M_0, M\right)\right] \\
& + \left[s_\theta^4\frac{M^2}{p^2} + 1 - 9c_\theta^4 + c_\theta^2\left(2-9c_\theta^2\right)\frac{p^2}{M^2} + c_\theta^4\frac{p^4}{M^4}\right] \\
& \qquad\qquad \times\left[B_0\left(p^2; M, \xi_Z M_0\right) - B_0\left(p^2; M_0, M\right)\right] \\
& +2s_\theta^2\left(\frac{M^2}{p^2} + 8 - 5\frac{p^2}{M^2}\right)\left[B_0\left(p^2; 0, \xi M\right) - B_0\left(p^2; 0, M\right)\right] \\
& + \Bigg[\left(\xi_Z^2 - 1\right)\left(\xi_Z^2 + 1 + 2c_\theta^2\frac{p^2}{M^2}\right) + c_\theta^4\left(\xi^2-1\right)\left(\xi^2 + 1 + 2\frac{p^2}{M^2}\right)
\end{aligned}$$

$$
\begin{aligned}
&\qquad -2c_\theta^2\left(\xi_Z^2\xi^2-1\right)\Big]\left(\frac{M^2}{p^2}-1\right)B_0\left(p^2;\xi M,\xi_Z M_0\right)\\
&+\left(\xi_Z^2-1\right)\left[\left(\xi_Z^2+1-2c_\theta^2\right)\frac{M^2}{p^2}+2c_\theta^2\right]\left(1+\frac{p^2}{M^2}\right)B_0\left(p^2;M,\xi_Z M_0\right)\\
&+\left(\xi^2-1\right)\left[c_\theta^2\xi^2\left(\left(2-c_\theta^2\right)\frac{M^2}{p^2}+c_\theta^2\right)-\left(2-c_\theta^2\right)^2\frac{M^2}{p^2}\right.\\
&\qquad\left.+c_\theta^2\left(2-c_\theta^2\right)+2c_\theta^4\frac{p^2}{M^2}\right]B_0\left(p^2;M_0,\xi M\right)\\
&+2s_\theta^2\left(\xi^2-1\right)\left[\left(\xi^2+1\right)\frac{M^2}{p^2}+2\right]B_0\left(p^2;0,\xi M\right)\\
&+3s_\theta^2\left(\xi_A^2-1\right)\left[\left(1-\xi^2\frac{M^2}{p^2}\right)\left(1-\frac{p^2}{M^2}\right)B_0\left(p^2;0,\xi M\right)\right.\\
&\qquad\left.-\left(\frac{M^2}{p^2}+4-\frac{p^2}{M^2}\right)B_0\left(p^2;0,M\right)\right]\\
&+2c_\theta^2\left(s_\theta^2\frac{M^2}{p^2}+5c_\theta^2\right)\frac{1}{M^2}\left[A_0\left(\xi_Z M_0\right)-A_0\left(M_0\right)\right]\\
&+\frac{10}{M^2}\left[A_0\left(\xi M\right)-A_0\left(M\right)\right]\\
&+\left[2\left(\xi^2-1\right)\frac{M^2}{p^2}-c_\theta^2\left(\xi_Z^2-1\right)\left(\frac{M^2}{p^2}-1\right)\right]\frac{1}{M^2}A_0\left(\xi M\right)\\
&-c_\theta^2\left[c_\theta^2\left(\xi^2-1\right)\left(\frac{M^2}{p^2}-1\right)-2\left(\xi_Z^2-1\right)\frac{M^2}{p^2}\right]\frac{1}{M^2}A_0\left(\xi_Z M_0\right)\\
&-c_\theta^2\left(\xi^2-1\right)\left[\left(2-c_\theta^2\right)\frac{M^2}{p^2}+c_\theta^2\right]\frac{1}{M^2}A_0\left(M_0\right)\\
&-c_\theta^2\left(\xi_Z^2-1\right)\left(\frac{M^2}{p^2}+1\right)\frac{1}{M^2}A_0\left(M\right)\\
&-3s_\theta^2\frac{\xi_A^2-1}{p^2}\left[A_0\left(\xi M\right)+A_0\left(M\right)-\frac{p^2}{M^2}\left[A_0\left(\xi M\right)-A_0\left(M\right)\right]\right]\\
&+4c_\theta^2\left(\xi_Z^2-1\right)+4\left(\xi^2-1\right)+24s_\theta^2\left(\xi_A^2-1\right)\Big\}\\
&+2s_\theta^2\left(\xi_A^2-1\right)\left[M^2B_0\left(p^2;0,M\right)+A_0\left(M\right)-M^2\right]
\end{aligned}
$$

U gauge.

The number of total self energies in the U-gauge is very limited. Below the whole list is presented. The following auxiliary parameters are used:

$$w = \frac{p^2}{M_W^2}\,, \quad z = \frac{p^2}{M_Z^2}\,, \quad h = \frac{p^2}{M_H^2}\,, \quad w_h = \frac{M_H^2}{M_W^2}\,, \quad h_W^{(i)} = (1-w_h)^i$$

$$\begin{aligned}
\frac{\Sigma^U_{WW}(p^2)}{M_W^2} = & \Bigg[-\left(\frac{1}{12c_W^4}+\frac{2}{3}\frac{1}{c_W^2}-\frac{3}{2}+\frac{2}{3}c_W^2+\frac{1}{12}c_W^4\right)\frac{1}{w} \\
& +\frac{2}{3}\left(\frac{1}{c_W^2}-4-4c_W^2+c_W^4\right)+\left(\frac{3}{2}+\frac{8}{3}c_W^2+\frac{3}{2}c_W^4\right)w \\
& +\frac{2}{3}c_W^2\left(1+c_W^2\right)w^2-\frac{1}{12}c_W^4w^3\Bigg]B_0\left(p^2;M_Z,M_W\right) \\
& -\frac{s_W^2}{6}\left(\frac{5}{w}+17-17w-5w^2\right)B_0\left(p^2;0,M_W\right) \\
& -\frac{1}{12}\left(\frac{w_h^{(2)}}{w}-10+2\,w_h+w\right)B_0\left(p^2;M_H,M_W\right) \\
& +\Bigg[\frac{1}{12}\left(\frac{1}{c_W^2}-2+c_W^2-c_W^4+w_h\right)\frac{1}{w}-2+\frac{1}{6}c_W^2-\frac{1}{12}c_W^4 \\
& +\frac{1}{12}\left(-10+c_W^2+c_W^4\right)w+\frac{1}{12}c_W^4w^2\Bigg]\frac{A_0\left(M_W\right)}{M_W^2} \\
& +\Bigg[-\frac{1}{12}\left(\frac{1}{c_W^2}+9-9c_W^2-c_W^4\right)\frac{1}{w}-\frac{1}{12}-\frac{7}{6}c_W^2-\frac{3}{4}c_W^4 \\
& +\frac{1}{12}\left(c_W^2-9c_W^4\right)w+\frac{1}{12}c_W^4w^2\Bigg]\frac{A_0\left(M_Z\right)}{M_W^2} \\
& +\frac{1}{12}\left(\frac{1}{w}-\frac{1}{h}-2\right)\frac{A_0\left(M_H\right)}{M_W^2} \\
& -\frac{1}{6}\left(\frac{1}{c_W^2}+22+c_W^2+c_W^4+w_h\right)-\frac{1}{9}\left(2+3c_W^2+\frac{7}{2}c_W^4\right)w \\
& -\frac{1}{9}\left(1+\frac{3}{2}c_W^2+\frac{5}{2}c_W^4\right)w^2-\frac{1}{18}c_W^4w^3
\end{aligned}$$

$$S_{ZZ} = \frac{g^2}{16\pi^2 c_\theta^2}\Sigma_{ZZ}(p^2), \quad S_{ZA} = \frac{g^2 s_\theta}{16\pi^2 c_\theta}\Sigma_{ZA}(p^2), \quad S_{AA} = \frac{g^2}{16\pi^2}\Sigma_{AA}(p^2)$$

$$\begin{aligned}
\frac{\Sigma^U_{ZZ}(p^2)}{M_W^2} &= c_W^4\left(-4+\frac{17}{3}w+\frac{4}{3}w^2-\frac{w^3}{12}\right)B_0\left(p^2;M_W,M_W\right) \\
&\quad +\frac{1}{12}\left[-\left(\frac{1}{c_W^4}-2\frac{w_h}{c_W^2}+w_h^2\right)\frac{1}{w}+\frac{10}{c_W^2}-2w_h-w\right] \\
&\qquad\qquad \times B_0\left(p^2;M_H,M_Z\right) \\
&\quad +c_W^2\left(-4-\frac{4}{3}w+\frac{w^2}{6}\right)\frac{A_0\left(M_W\right)}{M_Z^2} \\
&\quad +\frac{1}{12c_W^2}\left(\frac{1}{h}-\frac{1}{z}+1\right)\frac{A_0\left(M_Z\right)}{M_Z^2} \\
&\quad -\frac{1}{12c_W^2}\left(\frac{1}{h}-\frac{1}{z}+2\right)\frac{A_0\left(M_H\right)}{M_Z^2} \\
&\quad -\left[\frac{1}{6c_W^2}+4c_W^4+\frac{w_h}{6}+\left(\frac{1}{18}+\frac{4}{3}c_W^4\right)w+\frac{5}{9}c_W^4w^2+\frac{1}{18}c_W^4w^3\right]
\end{aligned}$$

$$\Sigma^U_{AA}(p^2) = s_W^2 p^2 \Pi^U_{\gamma\gamma}(p^2), \qquad \Sigma^U_{ZA}(p^2) = c_W^2 p^2 \Pi^U_{\gamma\gamma}(p^2)$$

$$\begin{aligned}
\Pi^U_{\gamma\gamma}(p^2) &= \frac{1}{w}\left[\left(-4+\frac{17}{3}w+\frac{4}{3}w^2-\frac{w^3}{12}\right)B_0\left(p^2;M_W,M_W\right)\right. \\
&\quad \left.+\left(-4-\frac{4}{3}w+\frac{w^2}{6}\right)\frac{A_0\left(M_W\right)}{M_W^2}-4-\frac{4}{3}w-\frac{5}{9}w^2-\frac{w^3}{18}\right]
\end{aligned}$$

$$\begin{aligned}
\frac{\Sigma^U_{HH}(p^2)}{M_W^2} &= \left(3+w+\frac{w^2}{4}\right)B_0\left(p^2;M_W,M_W\right)+\frac{9}{8}w_h^2B_0\left(p^2;M_H,M_H\right) \\
&\quad +\frac{1}{2c_W^4}\left(3+z+\frac{z^2}{4}\right)B_0\left(p^2;M_Z,M_Z\right)+\left(3-\frac{w}{2}\right)\frac{A_0\left(M_W\right)}{M_W^2} \\
&\quad +\left(\frac{3}{2c_\theta^2}-\frac{w}{4}\right)\frac{A_0\left(M_Z\right)}{M_W^2}+\frac{3}{4}w_h\frac{A_0\left(M_H\right)}{M_W^2}
\end{aligned}$$

NB: the mixing, $\Sigma^U_{ZA}(p^2) \propto \Pi^U_{\gamma\gamma}(p^2)$ (holds only in the U-gauge!)

Fermionic components of bosonic self-energies

$$\Pi^{\rm fer}_{\gamma\gamma}(p^2) = 4\sum_f c_f Q_f^2 B_f\left(p^2;m_f,m_f\right)$$

$$\Sigma^{\rm fer}_{ZA}(p^2) = \sum_f c_f Q_f v_f p^2 B_f\left(p^2;m_f,m_f\right)$$

$$\Sigma^{\rm fer}_{ZZ}(p^2) = \sum_f c_f\left[\left(v_f^2+a_f^2\right)p^2 B_f\left(p^2;m_f,m_f\right) - 2a_f^2 m_f^2 B_0\left(p^2;m_f,m_f\right)\right]$$

$$\Sigma^{\rm fer}_{WW}(p^2) = \sum_{f=d} c_f p^2 B_f\left(p^2;m_{f'},m_f\right) + \sum_f c_f m_f^2 B_1\left(p^2;m_{f'},m_f\right)$$

$$\Sigma^{\rm fer}_{HH}(p^2) = \sum_f c_f\frac{m_f^2}{M_W^2}\left[A_0\left(m_f\right) - \frac{p^2+4m_f^2}{2}B_0\left(p^2;m_f,m_f\right)\right]$$

The fermionic component of the $H-V$ transition vanishes

$$\propto\left[B_0\left(p^2;m_f,m_f\right)+2B_1\left(p^2;m_f,m_f\right)\right]p_\mu$$

The definition of the function B_f

$$B_f\left(p^2;m_{f'},m_f\right) = 2\left[B_{21}\left(p^2;m_{f'},m_f\right)+B_1\left(p^2;m_{f'},m_f\right)\right]$$

Pole and finite parts of the B_{ij} functions

$$B_{ij}\left(p^2;m_1,m_2\right) = c_{ij}\left(\frac{1}{\bar\varepsilon} - \ln\frac{M_W^2}{\mu^2}\right) + B^F_{ij}\left(p^2;m_1,m_2\right)$$

$$c_0 = 1\,,\qquad c_1 = -\frac{1}{2}\,,\qquad c_{21} = \frac{1}{3}$$

For equal masses $m_1 = m_2 = m_f$

$$p^2 B^F_f\left(p^2;m_f,m_f\right) = \frac{p^2}{9} + \frac{2m_f^2}{3}\ln\frac{m_f^2}{M_W^2} + \frac{1}{3}\left(2m_f^2-p^2\right)B^F_0\left(p^2;m_f,m_f\right)$$

and

$$B_1\left(p^2;m_f,m_f\right) = -\frac{1}{2}B_0\left(p^2;m_f,m_f\right)$$

$$B^F_0\left(p^2;m_f,m_f\right) = 2 - \ln\frac{m_f^2}{M_W^2} - \beta\ln\frac{\beta+1}{\beta-1}\,,\qquad \beta = \sqrt{1+4\frac{m_f^2}{p^2}}$$

Asymptotic behavior of B functions and self-energies

Realistic situation: all fermions but top quark are massless.
More approximation:

$$|p^2| << m_t^2$$

i.e. m_t is the largest and the only scale

	$m_{f'} = m_t$, $m_f = m_t$	$m_{f'} = 0$, $m_f = m_t$	$m_{f'} = m_t$, $m_f = 0$
$B_0^F(p^2; m_{f'}, m_f) \to$	$-\ln\frac{m_t^2}{\mu^2}$	$1 - \ln\frac{m_t^2}{\mu^2}$	the same
$B_1^F(p^2; m_{f'}, m_f) \to$	$\frac{1}{2}\ln\frac{m_t^2}{\mu^2}$	$\frac{1}{2}\ln\frac{m_t^2}{\mu^2} - \frac{1}{4}$	$\frac{1}{2}\ln\frac{m_t^2}{\mu^2} - \frac{3}{4}$
$B_f^F(p^2; m_{f'}, m_f) \to$	$\frac{1}{2}\ln\frac{m_t^2}{\mu^2}$	$\frac{1}{3}\ln\frac{m_t^2}{\mu^2} - \frac{5}{18}$	the same

Using this table, one derives for ZZ and WW self-energies

$$\begin{aligned} \Sigma^{\text{fer}}_{ZZ}(0) &= \frac{3}{2}m_t^2 \ln\frac{m_t^2}{\mu^2} \\ \Sigma^{\text{fer}}_{WW}(0) &= \frac{3}{2}m_t^2 \left(\ln\frac{m_t^2}{\mu^2} - \frac{1}{2}\right) \end{aligned}$$

Veltman's ρ parameter (original)

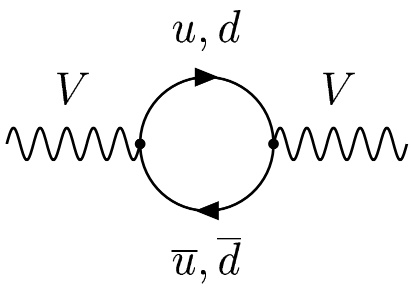
$$\Delta\rho = \frac{1}{M_W^2}\left[\Sigma^{\text{fer}}_{WW}(0) - \Sigma^{\text{fer}}_{ZZ}(0)\right] \approx -\frac{3}{4}\frac{m_t^2}{M_W^2}$$

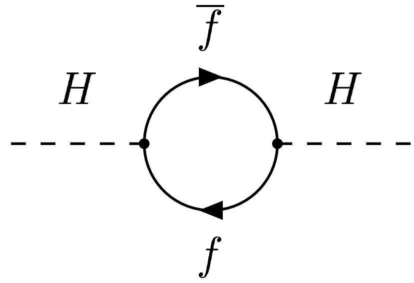
More relevant quantity is made of *complete* self-energies

$$\Delta\rho = \frac{1}{M_W^2}\left[\Sigma_{WW}(M_W^2) - \Sigma_{ZZ}(M_Z^2)\right] \approx -\frac{3}{4}\frac{m_t^2}{M_W^2}$$

It is the gauge invariant but ultraviolet divergent object.

Ultraviolet behaviour of fermionic components of bosonic self-energies
(more physics and ... politics)
Consider two diagrams:





Common expression

$$\Sigma \;\propto\; \frac{Tr[(i\not{q} + m_{f'})\,\Gamma_1\,(i\not{p} + i\not{q} + m_f)\,\Gamma_2]}{\left(q^2 + m_{f'}^2\right)[(q+p)^2 + m_f^2]}$$

Vector case, e.g. $\Gamma_1 = \gamma_\mu,\ \Gamma_2 = \gamma_\nu$

$$(\Sigma_V)_{\mu\nu} \;\propto\; 4\frac{\delta_{\mu\nu}[q\,(p+q) + m_f^2] - (q_\mu p_\nu + q_\nu p_\mu) - 2q_\mu q_\nu}{\left(q^2 + m_f^2\right)[(q+p)^2 + m_f^2]}$$

Scalar case, $\Gamma_1 = \Gamma_2 = 1$

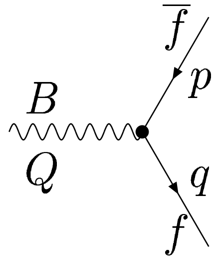
$$\Sigma_S \;\propto\; 4\frac{q^2 - p\cdot q - m_f^2}{\left(q^2 + m_f^2\right)[(q+p)^2 + m_f^2]}$$

Leading UV divergences

$$\begin{aligned}(\Sigma_V)_{\mu\nu} &\propto 4\frac{\delta_{\mu\nu} q^2 - 2q_\mu q_\nu}{\left(q^2 + m_f^2\right)[(q+p)^2 + m_f^2]} \\ &= \delta_{\mu\nu} i\pi^{\frac{n}{2}}\frac{1}{\Gamma(\alpha)}\Gamma\left(1 - \frac{n}{2}\right)\left(m^2 - p^2\right)^{\frac{n}{2}-1}\frac{1}{2}(n-2) \to \Gamma\left(2 - \frac{n}{2}\right) \\ \Sigma_S &\propto 4\frac{q^2}{\left(q^2 + m_f^2\right)[(q+p)^2 + m_f^2]} \\ &= i\pi^{\frac{n}{2}}\frac{1}{\Gamma(\alpha)}\Gamma\left(1 - \frac{n}{2}\right)\left(m^2 - p^2\right)^{\frac{n}{2}-1}\frac{n}{2} \to \Gamma\left(1 - \frac{n}{2}\right)\end{aligned}$$

Naturallness, SUSY, LHC and all that...

Calculation of decay rates in the Born approximation



PDG convention

$$d\Gamma = \frac{(2\pi)^4}{2M} \overline{\sum_{\text{spins}}} \mid \mathcal{M} \mid^2 d\Phi_2$$

Has to be redefined in order to generalize to n dimensions

$$d\Gamma = \frac{1}{2M} \overline{\sum_{\text{spins}}} \mid \mathcal{M} \mid^2 d\Phi_2$$

The two-body phase space

$$\Phi_2 = (2\pi)^4 \int \frac{d^3p}{(2\pi)^3 2p_0} \int \frac{d^3q}{(2\pi)^3 2q_0} \delta(Q - p - q)$$

Calculation of Φ_2

$$\begin{aligned}
\Phi_2 &= \frac{1}{(2\pi)^2} \int \frac{d^3p}{2p_0} \int d^4q\, \delta^+\left(q^2 + m_f^2\right) \delta(Q - p - q) \\
&\quad \left[\text{with} \quad \delta^+\left(p^2 + m_f^2\right) = \theta(p_0)\, \delta\left(p^2 + m_f^2\right)\right] \\
&= \frac{1}{(2\pi)^2} \int \frac{\mid \vec{p} \mid^2 d \mid \vec{p} \mid}{2p_0} d\Omega_p\, \delta^+\left[(Q - p)^2 + m_f^2\right] \\
&\quad \left[\text{using} \quad \mid \vec{p} \mid d \mid \vec{p} \mid = p_0 dp_0 \quad \text{and} \quad d\Omega_p \to 4\pi\right] \\
&= \frac{1}{2\pi} \int \mid \vec{p} \mid dp_0\, \delta\left(-M^2 + 2Mp_0\right) = \frac{1}{2\pi} \frac{\mid \vec{p} \mid}{2M}
\end{aligned}$$

Using: $p_0 = \frac{M}{2}, \quad \mid \vec{p} \mid = \sqrt{\frac{M^2}{4} - m_f^2}, \quad \beta_f(M) = \frac{\mid \vec{p} \mid}{p_0} = \sqrt{1 - \frac{4m_f^2}{M^2}}$

$$\boxed{\Phi_2 = \frac{1}{8\pi} \beta_f(M)}$$

Calculation of $\overline{\sum_{\text{spins}}} \mid \mathcal{M} \mid^2$ for three decays: $V,\ Z,\ H$.

Axial-vector case:

$$\begin{aligned}
\overline{\sum_{\text{spins}}} \mid \mathcal{M} \mid^2 &= \frac{1}{3}\left(\delta_{\mu\nu} + \frac{Q_\mu Q_\nu}{M^2}\right) \overline{\sum_{\text{spins}}} \mathcal{M}_\mu^{\text{Born}} \left(\mathcal{M}_\nu^{\text{Born}}\right)^+ \\
\mathcal{M}_\mu^{\text{Born}} &= i f \bar{u}(q) \gamma_\mu (v_f + a_f \gamma_5) v(p) \\
\left(\mathcal{M}_\mu^{\text{Born}}\right)^+ &= i f \bar{v}(p) \gamma_\nu (v_f + a_f \gamma_5) u(q)
\end{aligned}$$

where the coupling constant

$$f = \begin{cases} e & \text{for} \quad V = \text{heavy photon} \\ \dfrac{g}{2c_\theta} & \text{for} \quad V = Z \end{cases}$$

For non-polarized fermions

$$\overline{\sum_{\text{spins}}} u(q) \bar{u}(q) = -i\not{q} + m_f\ , \qquad \overline{\sum_{\text{spins}}} v(p) \bar{v}(p) = -i\not{p} - m_f$$

and

$$\overline{\sum_{\text{spins}}} \mid \mathcal{M} \mid^2 = \begin{cases} \dfrac{4}{3} e^2 M_V^2 \left(1 + 2\dfrac{m_f^2}{M_V^2}\right) & \text{for} \quad V = \text{heavy photon} \\ \dfrac{1}{3}\dfrac{g^2}{c_\theta^2} M_Z^2 \left[\left(v_f^2 + a_f^2\right)\left(1 + 2\dfrac{m_f^2}{M_Z^2}\right) - 6a_f^2 \dfrac{m_f^2}{M_Z^2}\right] & \text{for} \quad V = Z \end{cases}$$

Scalar case:

$$\begin{aligned}
\overline{\sum_{\text{spins}}} \mid \mathcal{M} \mid^2 &= \overline{\sum_{\text{spins}}} \mathcal{M}^{\text{Born}} \left(\mathcal{M}^{\text{Born}}\right)^+ \\
\mathcal{M}_\mu^{\text{Born}} &= -\frac{m_f}{2M_W} \bar{u}(q) v(p)\,, \qquad \left(\mathcal{M}^{\text{Born}}\right)^+ = -\frac{m_f}{2M_W} \bar{v}(p) u(q)
\end{aligned}$$

and

$$\overline{\sum_{\text{spins}}} \mid \mathcal{M} \mid^2 = \frac{g^2 m_f^2 M_H^2}{2M_W^2} \beta_f^2 (M_H)$$

List of answers for partial widths

$$\Gamma\left(V \to f\overline{f}\right) = \frac{e^2 M_V}{12\pi}\,\beta_f\left(M_V\right)\left(1+2\frac{m_f^2}{M_V^2}\right)$$

$$\Gamma\left(Z \to f\overline{f}\right) = 4\Gamma_0\,\beta_f\left(M_Z\right)\left[\left(v_f^2+a_f^2\right)\left(1+2\frac{m_f^2}{M_Z^2}\right)-6a_f^2\frac{m_f^2}{M_Z^2}\right]$$

$$\Gamma\left(H \to f\overline{f}\right) = \frac{G_F m_f^2 M_H}{4\sqrt{2}\pi}\,\beta_f^3\left(M_H\right)$$

here one used $\dfrac{G_F}{\sqrt{2}} = \dfrac{g^2}{8M_W^2}$, $\quad \dfrac{M_W^2}{M_Z^2} = c_W^2$, $\quad \Gamma_0 = \dfrac{M_Z^3}{24\sqrt{2}\pi}$

Calculation of partial widths through self-energy functions

$$S_{\gamma\gamma} = \frac{e^2}{16\pi^2}\left[-M_V^2 4B_f\left(-M_V^2; m_f, m_f\right)\right]$$

$$S_{ZZ} = \frac{g^2}{16\pi^2 c_\theta^2}\Big[-\left(v_f^2+a_f^2\right)M_Z^2 B_f\left(-M_Z^2; m_f, m_f\right)$$
$$-2a_f^2 m_f^2 B_0\left(-M_Z^2; m_f, m_f\right)\Big]$$

$$S_{HH} = \frac{g^2}{16\pi^2}\left[\frac{m_f^2}{M_W^2}\left(A_0\left(m_f\right)-\frac{-M_H^2+4m_f^2}{2}B_0\left(-M_H^2; m_f, m_f\right)\right)\right]$$

Recall

$$B_f\left(-M^2; m_f, m_f\right) = 2\left[B_{21}\left(-M^2; m_f, m_f\right)+B_1\left(-M^2; m_f, m_f\right)\right]$$

Taking imaginary parts:

$$\begin{aligned}
\mathrm{Im}\,A_0\left(m_f\right) &= 0\\
\mathrm{Im}\,B_1\left(-M^2; m_f, m_f\right) &= -\frac{1}{2}\mathrm{Im}\,B_0\left(-M^2; m_f, m_f\right)\\
\mathrm{Im}\,B_{21}\left(-M^2; m_f, m_f\right) &= \frac{1}{3}\left(1-\frac{m_f^2}{M_Z^2}\right)\mathrm{Im}\,B_0\left(-M^2; m_f, m_f\right)\\
\mathrm{Im}\,B_0\left(-M^2; m_f, m_f\right) &= \pi\beta_f\left(M\right)
\end{aligned}$$

$$\boxed{\mathrm{Im}\,S_{BB} = M_B\Gamma\left(B \to f\overline{f}\right)}$$

Dispersion relation for $\Pi\left(p^2\right)$:

$$\Pi\left(p^2\right) = \frac{6m_f^2+p^2}{9p^2} + \frac{2}{3p^2}A_0\left(m_f\right) - \frac{p^2-2m_f^2}{3p^2}B_0\left(p^2;m_f,m_f\right)$$

with

$$\begin{aligned} B_0\left(p^2;m_f,m_f\right) &= \frac{1}{\bar{\varepsilon}} - \ln\frac{m_f^2}{\mu^2} + B_0^F\left(p^2;m_f,m_f\right) \\ B_0^F\left(p^2;m_f,m_f\right) &= \int_0^1 dx \ln\frac{p^2x\left(1-x\right)+m_f^2}{m_f^2} = 2 - \beta\ln\frac{\beta+1}{\beta-1} \\ \beta^2 &= 1+4\frac{m_f^2}{p^2-i\epsilon} \end{aligned}$$

The renormalized vacuum polarization ($p^2=-s$)

$$\Pi^{\text{ren}}(s) = \Pi(s) - \Pi(0) = \frac{1}{9} - \frac{1}{3}\left(1+2\frac{m_f^2}{s}\right)B_0^F\left(-s;m_f,m_f\right)$$

Its imaginary part

$$\operatorname{Im}\Pi^{\text{ren}}(s) = -\frac{1}{3}\left(1+2\frac{m_f^2}{s}\right)\operatorname{Im}B_0^F\left(-s;m_f,m_f\right) = -\frac{1}{3}\left(1+2\frac{m_f^2}{s}\right)\pi\beta_f$$

Compute *dispersion integral*

$$\frac{s}{\pi}\int_{4m_f^2}^{\infty} d\tau \frac{\operatorname{Im}\Pi^{\text{ren}}(\tau)}{\tau\left(\tau-s-i\epsilon\right)} = -\frac{1}{3}\int_{4m_f^2}^{\infty}\frac{s d\tau}{\tau\left(\tau-s-i\epsilon\right)}\left(1+2\frac{m_f^2}{\tau}\right)\sqrt{1-\frac{4m_f^2}{\tau}}$$

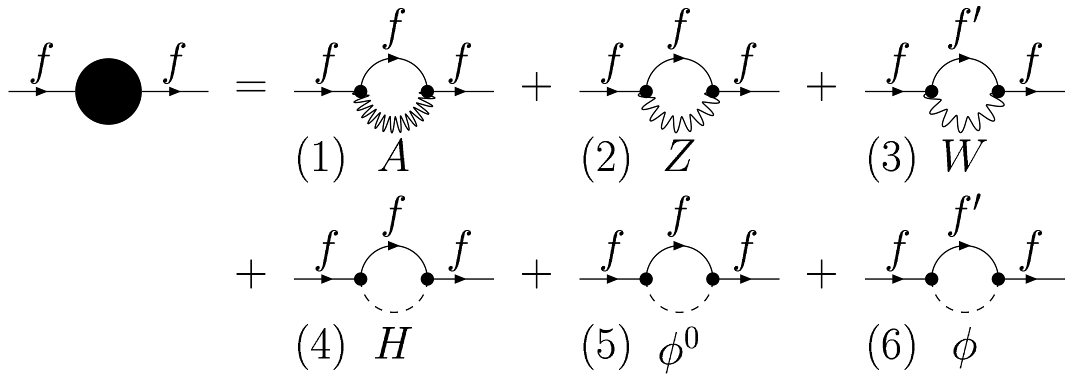
$$= -\frac{1}{3}\left(1+2\frac{m_f^2}{s}\right)\int_{4m_f^2}^{\infty}\frac{s d\tau}{\tau\left(\tau-s-i\epsilon\right)}\sqrt{1-\frac{4m_f^2}{\tau}} + \frac{2m_f^2}{3}\int_{4m_f^2}^{\infty}\frac{d\tau}{\tau^2}\sqrt{1-\frac{4m_f^2}{\tau}}$$

$$\int_{4m_f^2}^{\infty}\frac{s d\tau}{\tau\left(\tau-s-i\epsilon\right)}\sqrt{1-\frac{4m_f^2}{\tau}} = \beta\ln\frac{\beta+1}{\beta-1} - 2, \quad \int_{4m_f^2}^{\infty}\frac{d\tau}{\tau^2}\sqrt{1-\frac{4m_f^2}{\tau}} = \frac{1}{6m_f^2}$$

Substituting last two integrals, we see an **identity**

$$\boxed{\Pi^{\text{ren}}(s) = \frac{s}{\pi}\int_{4m_f^2}^{\infty} d\tau \frac{\operatorname{Im}\Pi^{\text{ren}}(\tau)}{\tau\left(\tau-s-i\epsilon\right)}}$$

Fermion self-energies in the standard model



f' is the weak isospin partner of the f-fermion.

$$v_f = I_f^{(3)} - 2s_\theta^2 Q_f, \qquad a_f = I_f^{(3)}$$

Combinations of couplings,

$$\begin{aligned}
\sigma_f &= v_f + a_f, \qquad \sigma_f^{(2)} = v_f^2 + a_f^2, \qquad \sigma_f^i = \left(v_f + a_f\right)^i \\
\delta_f &= v_f - a_f, \qquad \delta_f^{(2)} = v_f^2 - a_f^2, \qquad \delta_f^i = \left(v_f - a_f\right)^i
\end{aligned}$$

Each self energy diagram containing a B-line, $\Sigma_B\left(\not p\right)$,

$$\Sigma_B\left(\not p\right) = (2\pi)^4\, i\frac{g^2}{16\,\pi^2} A_B\ , \quad \not p = p_\alpha\gamma_\alpha$$

There are six A_B-functions in R_ξ gauge, and only four in the U gauge.

$$\begin{aligned}
A_A^\xi &= s_\theta^2\, Q_f^2 \Big\{ i\not p\left[2B_1\left(p^2;m_f,0\right)+1\right] - 2m_f\left[2B_0\left(p^2;m_f,0\right)-1\right] \\
&\quad - \left(i\not p + m_f\right)\left(\xi_A^2-1\right)\left[B_0\left(p^2;m_f,0\right) + m_f\left(i\not p - m_f\right) b_1\left(p^2;m_f\right)\right]\Big\}
\end{aligned}$$

In the U-gauge,

$$\begin{aligned}
A_Z^U = -\frac{1}{4c_\theta^2} \Bigg\{ & i\not p\left(\sigma_f^{(2)} + 2v_f a_f\gamma_5\right)\left[\frac{p^2+m_f^2}{M_0^2} B_1\left(p^2;M_0,m_f\right) + A_{\rm w}^U\left(p^2;M_0,m_f\right)\right] \\
& + m_f\delta_f^{(2)}\left[3B_0\left(p^2;M_0,m_f\right) + \frac{1}{M_0^2}A_0\left(m_f\right) - 2\right]\Bigg\}
\end{aligned}$$

$$A_W^U = -\frac{1}{4} i\not p\left(1+\gamma_5\right)\left\{\frac{p^2+m_{f'}^2}{M^2} B_1\left(p^2;M,m_f\right) + A_{\rm w}^U\left(p^2;M,m_f\right)\right\}$$

with the auxiliary function

$$A_{\rm w}^U\left(p^2;M,m\right) = 2B_1\left(p^2;M,m\right) + B_0\left(p^2;M,m\right) + \frac{1}{M^2}A_0(m) - 1$$

The standard model vertices: $V, S \to f_1 \overline{f}_2$

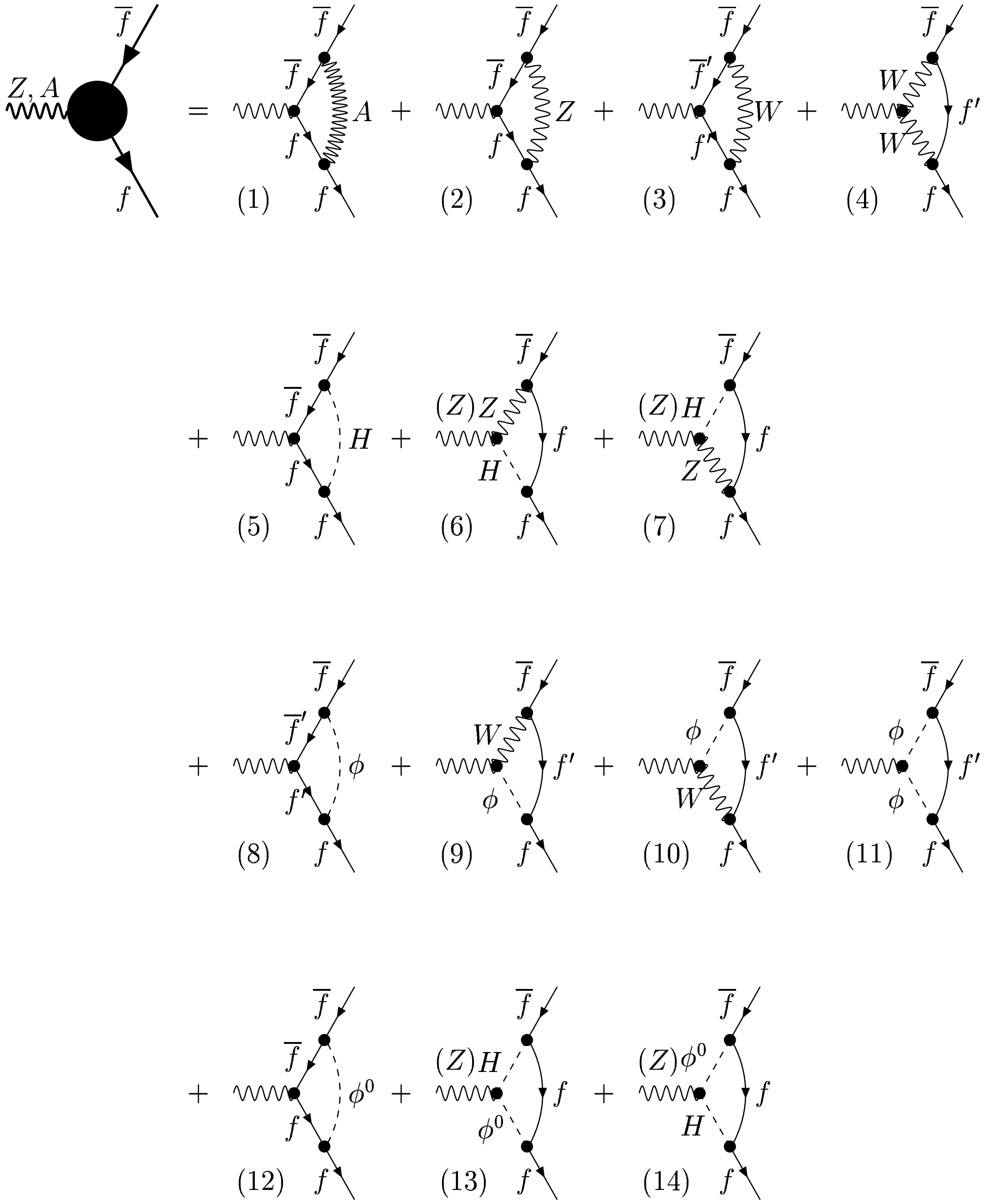


Figure 10: $(Z, A) \to f\overline{f}$ vertices. Symbol (Z) in some graphs indicates that given diagram contributes only to Z vertex.

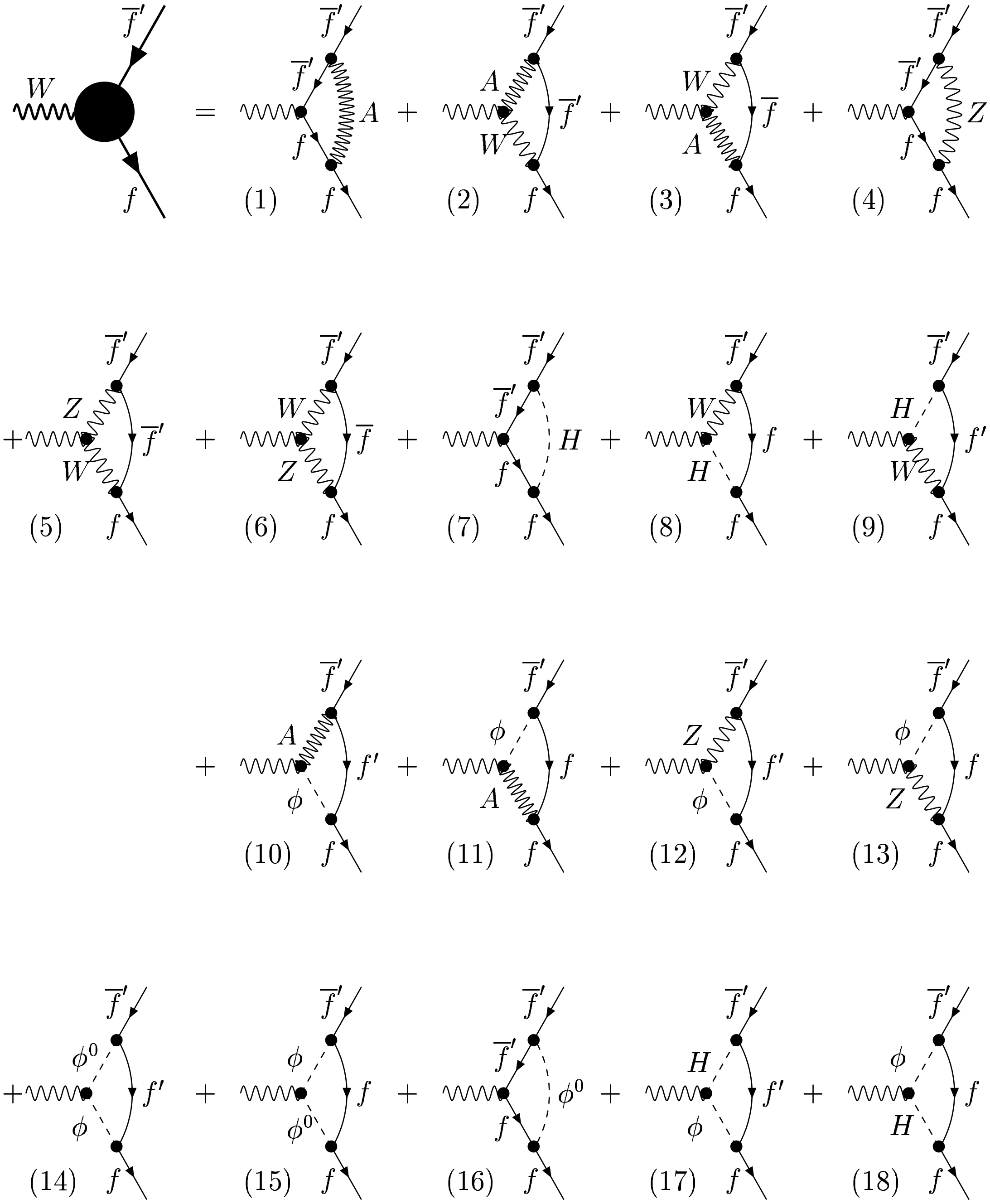


Figure 11: $W \to f\overline{f}'$ vertices.

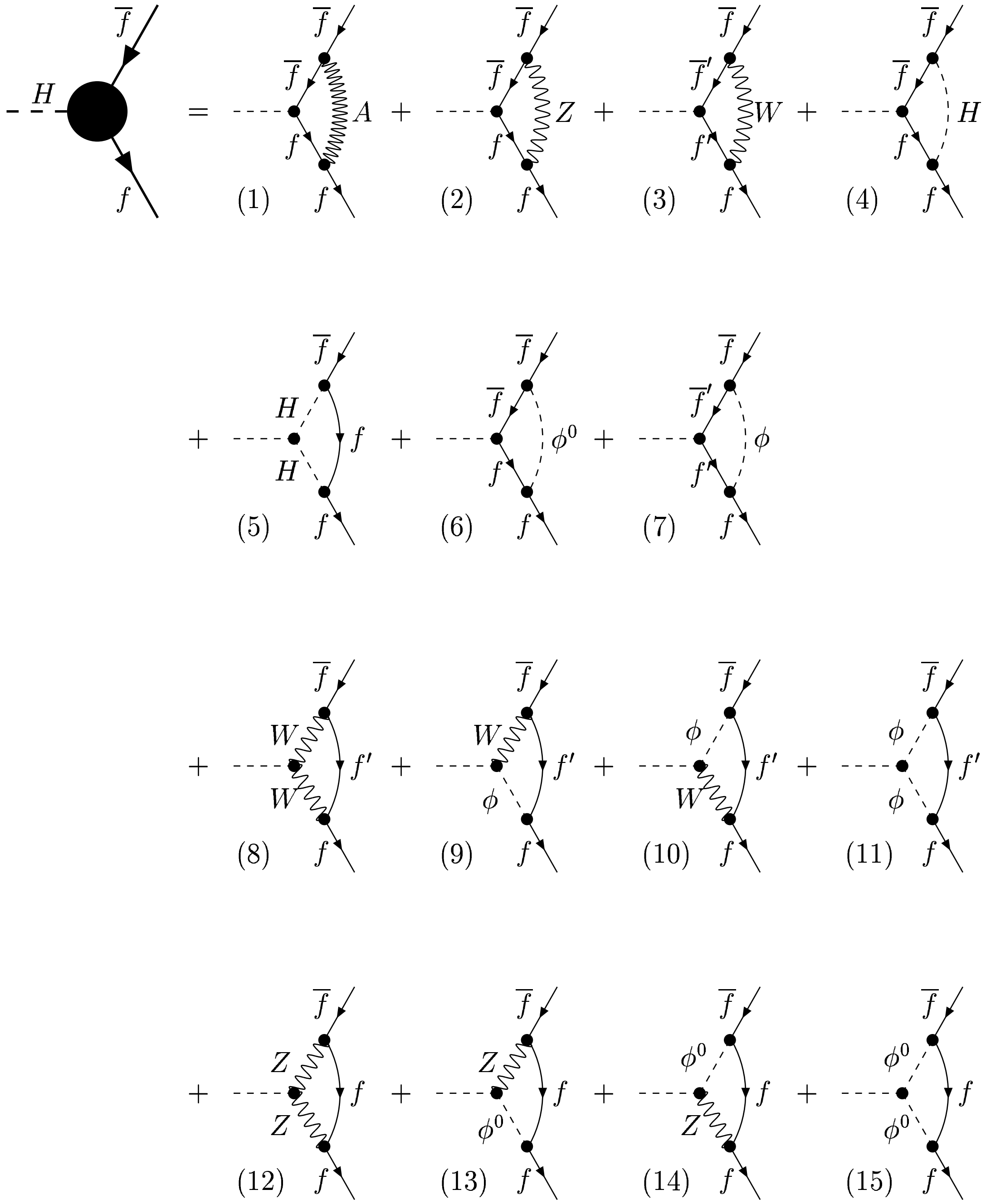


Figure 12: $H \to f\overline{f}$ vertices.

Example of a cluster

The *non-abelian* diagrams Fig. 10.(4,9-11) with virtual (W,ϕ) exchange, $V_\mu^{Wn}(Q^2)$. All the diagram 10.(9-11) contribute in our approximation (since $f'=t$ and $m_{f'}$ can't be neglected).

$\gamma f\overline{f}$ vertex

$$V_\mu^{Wn}\left(Q^2\right) = (2\pi)^4\, i\frac{ig^3}{16\pi^2}\frac{s_\theta}{2}\left(-I_f^{(3)}\right)\gamma_\mu\left(1+\gamma_5\right)F_{Wn}^{g}\left(Q^2\right)$$

$Zf\overline{f}$ vertex

$$V_\mu^{Wn}\left(Q^2\right) = (2\pi)^4\, i\frac{ig^3}{16\pi^2}\frac{c_\theta}{2}\left(-I_f^{(3)}\right)\gamma_\mu\left(1+\gamma_5\right)F_{Wn}^{g}\left(Q^2\right)$$

<u>U gauge</u>

$$\begin{aligned}
F_{Wn}^{U}\left(Q^2\right) = &\left[-\left(1-\frac{m_{f'}^2}{M^2}\right)^2\left(2+\frac{m_{f'}^2}{M^2}\right)\frac{M^2}{Q^2}+4-\frac{5}{2}\frac{m_{f'}^2}{M^2}+2\frac{m_{f'}^4}{M^4}-\frac{m_{f'}^6}{2M^6}\right.\\
&\left.-\frac{m_{f'}^2}{M^2}\left(2-\frac{m_{f'}^2}{2M^2}\right)\frac{Q^2}{M^2}\right]M^2C_0\left(0,0,Q^2;M,m_{f'},M\right)\\
&-\left[\frac{2}{3}-\frac{m_{f'}^2}{2M^2}-\left(\frac{3}{2}-\frac{m_{f'}^2}{4M^2}\right)\frac{Q^2}{M^2}+\frac{Q^4}{12M^4}\right]B_0\left(Q^2;M,M\right)\\
&-\left[\left(1-\frac{m_{f'}^2}{M^2}\right)\left(2+\frac{m_{f'}^2}{M^2}\right)\frac{M^2}{Q^2}-3+\frac{3}{2}\frac{m_{f'}^2}{M^2}-\frac{m_{f'}^4}{2M^4}\right]\\
&\times\left[B_0\left(Q^2;M,M\right)-B_0\left(0;m_{f'},M\right)\right]\\
&-\left(\frac{2}{3}-\frac{Q^2}{6M^2}\right)\frac{1}{M^2}A_0\left(M\right)-\frac{1}{M^2}A_0\left(m_{f'}\right)\\
&-\frac{2}{3}-\frac{m_{f'}^2}{2M^2}-\left(\frac{4}{9}-\frac{m_{f'}^2}{4M^2}\right)\frac{Q^2}{M^2}-\frac{Q^4}{18M^4}
\end{aligned}$$

This vertex, as well as the *abelian* diagrams Fig. 10.(3,8) with virtual (W,ϕ) exchange, are one more source of m_t^2/M_W^2 enhanced terms.

```
#procedure CalcDirBoxU(iu,id,fu,fd)
#call direct{'iu'|'id'|'fu'|'fd'}
#call prediracizing{dummy}
.sort
#call prereduction{dummy}
.sort
#call reduction{dummy}
#call scalprod{dummy}
.sort
#call diracizing{dummy}
#call scalprod{dummy}
#call diraceq{'iu'|'id'|'fu'|'fd'}
.sort
#call epsilon{dummy}
#call scalprod{dummy}
#call sing{dummy}
.sort
#call diraceq{'iu'|'id'|'fu'|'fd'}
#call extmomsshell{'iu'|'id'|'fu'|'fd'}
#call masses{dummy}
#call equalizing{dummy}
#call twohel{dummy}
.sort
#call scalarizingd2{dummy}
#call scalarizingd1{dummy}
#call scalarizingc2{dummy}
#call scalarizingc1{dummy}
#call scalarizingb{dummy}
#call extmomsshell{'iu'|'id'|'fu'|'fd'}
#call extmomsshellarg{'iu'|'id'|'fu'|'fd'}
#call masses{dummy}
#endprocedure
```

Summary of Level 4

1) Standard Model, its Fields and Lagrangian

Feynman Rules $\rightarrow$ *building* of diagrams

2) Regularization, N-point functions

A, B, C, D-functions $\rightarrow$ *calculation* of diagrams

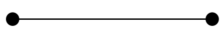
3) Groups of diagrams, *building blocks:*
Tadpoles $\rightarrow$ made of one point functions
Self-energies $\rightarrow$ two and one point functions

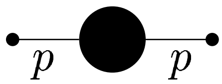
ρ-parameter
m_t^2-enchanced terms
problem of quadratic divergences

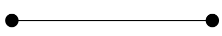
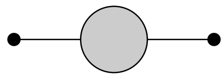
Vertices $\rightarrow$ 3,2,1 point functions
Boxes (direct / crossed) $\rightarrow$ 4,3,2,1 functions

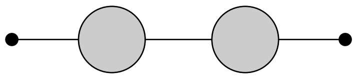
4) Approaching calculation of *amplitudes* for physical observables,
inevitability of renormalization.

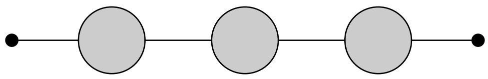
Dyson resummation

 $= \frac{1}{(2\pi)^4 i}\,\frac{1}{(p^2+M^2)}$

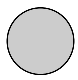


$=$  $+$ 

$+$ 

$+$ 

$+ \quad \cdots$

$= \frac{1}{(2\pi)^4 i}\,\frac{1}{\left[\, p^2+M^2 \;-\; \frac{1}{(2\pi)^4 i}\right.}$  $\left.\right]$

In case of conventional QED

$$S_{\mu\nu} \;=\; i\pi^2 e^2\left(p^2\delta_{\mu\nu}-p_\mu p_\nu\right)4\Pi\left(p^2\right)$$

$$\frac{1}{(2\pi)^4 i}\,\frac{\delta_{\mu\nu}}{p^2} \;\to\; \frac{1}{(2\pi)^4 i}\,\frac{\delta_{\mu\nu}}{p^2}\,\frac{1}{1-\frac{e^2}{4\pi^2}\Pi\left(p^2\right)}$$

$$\Pi(p^2) \;=\; 2\left[B_{21}\left(p^2;m,m\right)+B_1\left(p^2;m,m\right)\right]$$

Renormalization in QED

QED describes the interaction of spin-$\frac{1}{2}$ particles with photons.
QED Lagrangian in Feynman gauge is

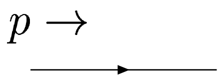
$$\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2} \left(\mathcal{C}^A\right)^2$$

$$-\sum_f \overline{\psi}_f \left(\not\partial - ieQ_f \not{A} + m_f\right) \psi_f$$

where

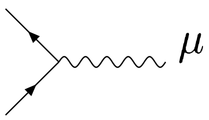
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \qquad \mathcal{C}^A = -\partial_\mu A_\mu$$

and the sum runs over fermion fields, f (charge eQ_f, and mass m_f)
The Feynman rules of QED:



$$\frac{1}{(2\pi)^4 i} \frac{-i\not{p} + m_f}{p^2 + m_f^2 - i\epsilon}$$

$$\frac{1}{(2\pi)^4 i} \frac{1}{p^2 + i\epsilon} \delta_{\mu\nu}$$



$$- \quad (2\pi)^4 i \ ieQ_f \gamma_\mu$$

On-shell renormalization in QED

QED Lagrangian is unambiguous at the tree level. Moving to higher orders, we assume that it is made of bare fields and parameters labelled with indices 0, and specify the renormalization constants for both fields — A_μ and ψ — and parameters — the mass m and the charge e

$$\begin{aligned} A^0_\mu &= Z_A^{1/2} A_\mu \ , \qquad \psi_0 = Z_\psi^{1/2} \psi \\ e_0 &= Z_e \, e \ , \qquad\quad m_0 = Z_m \, m = m + e^2 \delta m + \mathcal{O}\left(e^4\right) \\ Z_i &= 1 + e^2 \delta Z_i + \mathcal{O}\left(e^4\right) \end{aligned}$$

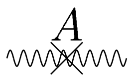
The Lagrangian can now be re-written, up to terms $\mathcal{O}\left(e^2\right)$

$$\mathcal{L}^{\mathrm{R}}_{\mathrm{QED}} = \mathcal{L}_{\mathrm{QED}} + \mathcal{L}_{\mathrm{ct}}$$

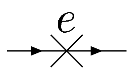
with a counter-term Lagrangian

$$\begin{aligned}
\mathcal{L}_{\mathrm{ct}} &= e^2\,\mathcal{L}^{(2)}_{\mathrm{ct}} + \mathcal{O}\left(e^4\right) \\
\mathcal{L}^{(2)}_{\mathrm{ct}} &= -\frac{1}{4}\,\delta Z_{A}\,F_{\mu\nu}F_{\mu\nu} - \frac{1}{2}\,\delta Z_{A}\,\left(\partial_\mu A_\mu\right)^2 - \delta Z_{\psi}\overline{\psi}\partial\!\!\!/\psi \\
&\quad - \left(\delta Z_{\psi} m + \delta m\right)\overline{\psi}\psi - i\left(\delta Z_e + \delta Z_{\psi} + \frac{1}{2}\delta Z_{A}\right) eA_\mu\overline{\psi}\gamma_\mu\psi
\end{aligned}$$

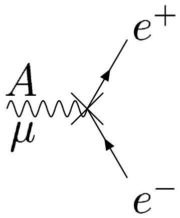
The counter-term Lagrangian generates a new set of Feynman rules



$$\rightarrow \qquad e^2\delta Z_{A}$$



$$\rightarrow \qquad -e^2\left(\delta Z_{\psi} i p\!\!\!/ + \delta Z_{\psi} m + \delta m\right)$$



$$\rightarrow \qquad -ie\gamma_\mu e^3\left(\delta Z_e + \delta Z_{\psi} + \frac{1}{2}\delta Z_{A}\right).$$

and we have to take into account contributions generated by both pieces

<u>The photon self-energy</u>

$$\begin{aligned}
S_{\mu\nu} &= i\pi^2 e^2\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right)4\Pi\left(p^2\right) \\
\Pi\left(p^2\right) &= 2\Big[B_{21}\left(p^2;m,m\right) + B_1\left(p^2;m,m\right)\Big] = -\frac{1}{3\bar{\varepsilon}} + B^{F}_{f}\left(p;m,m\right) \\
\Pi\left(0\right) &= \frac{1}{3}\left(-\frac{1}{\bar{\varepsilon}} + \ln\frac{m^2}{\mu^2}\right)
\end{aligned}$$

The $p_\mu p_\nu$ part does not contribute whenever one consider $S_{\mu\nu}$ as coupled to conserved fermionic currents. Therefore

$$S_{\mu\nu} = \Pi_0\,p^2\delta_{\mu\nu}\;, \qquad \Pi_0 = \left(2\pi\right)^4 i\,\frac{e^2}{4\pi^2}\Pi\left(p^2\right)$$

The Dyson re-summed (sometimes called dressed) photon propagator

$$D_{\mu\nu} = \frac{1}{(2\pi)^4 i} \frac{\delta_{\mu\nu}}{p^2} \frac{1}{1 + e^2 \delta Z_A - \frac{e^2}{4\pi^2} \Pi\left(p^2\right)}$$

Similarly, for the resummed electron propagator

$$\begin{aligned} S &= \frac{1}{(2\pi)^4 i} \Big\{ \left(1 + e^2 \delta Z_\psi\right) (i\not{p} + m) + e^2 \delta m \\ &\quad - \frac{1}{(2\pi)^4 i} \Big[\Sigma(im) + (i\not{p} + m) \Sigma_{\mathrm{WF}} + \mathcal{O}\left((i\not{p} + m)^2\right)\Big]\Big\}^{-1} \end{aligned}$$

The first two terms in the Taylor expansion of $\Sigma(\not{p})$ around the physical electron mass $i\not{p} = -m$ (sometimes called *subtraction point*)

$$\Sigma(\not{p}) = \Sigma(im) + (i\not{p} + m) \Sigma_{\mathrm{WF}} + \mathcal{O}\left((i\not{p} + m)^2\right)$$

where the coefficient of the linear term sometimes called wave function renormalization factor

$$\Sigma_{\mathrm{WF}} = \frac{\partial \Sigma(\not{p})}{\partial (i\not{p})}\Big|_{i\not{p} = -m}$$

By strightforward calculations in dimensional regularization

$$\begin{aligned} \Sigma(im) &= i\pi^2 e^2 m \left(-\frac{3}{\bar{\varepsilon}} + 3 \ln \frac{m^2}{\mu^2} - 4 \right) \\ \Sigma_{\mathrm{WF}} &= i\pi^2 e^2 \Big\{ 2B_1\left(-m^2; m, 0\right) + 1 \\ &\quad -4\, m^2 \Big[B_{1p}\left(-m^2; m, 0\right) + 2\, B_{0p}\left(-m^2; m, 0\right)\Big]\Big\} \\ &= i\pi^2 e^2 \left(-\frac{1}{\bar{\varepsilon}} + \frac{2}{\hat{\varepsilon}} + 3 \ln \frac{m^2}{\mu^2} - 4 \right) \end{aligned}$$

The one loop $e^+e^-\gamma$ vertex with both fermions on mass shell In terms of $V_1\left(Q^2; m, m\right)$, the γ_μ-part of the one-loop $e^+e^-\gamma$ vertex becomes

$$-(2\pi)^4\, i\, ie \left\{ 1 + e^2 \left[\delta Z_e + \frac{1}{2} \delta Z_A + \delta Z_\psi + \frac{1}{16\pi^2} V_1\left(Q^2; m, m\right) \right] \right\} \gamma_\mu$$

The essence of the on-mass-shell (OMS) renormalization scheme is to preserve the meaning of the original parameters of the Lagrangian.

First fixation condition

For the dressed photonic propagator, we require that its residue should be unchanged at the photonic mass shell, $p^2 = 0$, i.e.

$$e^2 \delta Z_A = \frac{e^2}{4\pi^2} \Pi(0)$$

This requirement guarantees that the wave function for external photonic lines does not change due to one loop radiative corrections and simultaneously fixes $e^2 \delta Z_A$

$$\delta Z_A = \frac{1}{12\pi^2} \left(-\frac{1}{\bar{\varepsilon}} + \ln \frac{m^2}{\mu^2} \right)$$

Second fixation condition

For the dressed electron propagator we require *residue =1* at the electron mass shell, $i\not{p} = -m$, i.e.

$$S = \frac{1}{(2\pi)^4 i \, (i\not{p} + m)}$$

This requirement preserves the external line electron wave function from being renormalized by one loop radiative corrections and simultaneosly fixes two more counter-terms

$$e^2 \delta m = \frac{\Sigma(im)}{(2\pi)^4 i}, \qquad e^2 \delta Z_\psi = \frac{\Sigma_{\mathrm{WF}}}{(2\pi)^4 i}$$

or

$$\delta m = \frac{m}{16\pi^2} \left(-\frac{3}{\bar{\varepsilon}} + 3 \ln \frac{m^2}{\mu^2} - 4 \right), \quad \delta Z_\psi = \frac{1}{16\pi^2} \left(-\frac{1}{\bar{\varepsilon}} + \frac{2}{\hat{\varepsilon}} + 3 \ln \frac{m^2}{\mu^2} - 4 \right)$$

Third fixation condition

For the one-loop corrected vertex we require it to be

$$-(2\pi)^4 i \, ie\gamma_\mu$$

at $Q^2 = 0$, which preserves the Thomson limit of the electric charge from being renormalized by one loop radiative corrections

$$\delta Z_e + \frac{1}{2} \delta Z_A + \delta Z_\psi + \frac{1}{16\pi^2} V_1(0; m, m) = 0$$

Substituting already fixed counter-term δZ_ψ, and the derived expression for $V_1(0;m,m)$, we observe famous QED Ward identity

$$\delta Z_\psi + \frac{1}{16\pi^2} V_1(0;m,m) \equiv 0$$

that fixes the last counter-term

$$\delta Z_e \equiv -\frac{1}{2}\delta Z_A$$

Now all the counter-terms in the Lagrangian are fixed and one may calculate any QED process at the one loop level.
The one loop and the counter-term contributions for any external on-shell line compensate each other identically (this is known as the principle of non-renormalizability for external lines).
For any $2 \to 2$ fermion process, at one loop level, we encounter only two building blocks:
1) the effective (running) electric charge, $e^2(p^2)$, entering photonic propagator

$$e^2 D_{\mu\nu} = \frac{e^2(p^2)}{(2\pi)^4 i}\frac{\delta_{\mu\nu}}{p^2}, \qquad e^2\left(p^2\right) = \frac{e^2}{1 - \frac{e^2}{4\pi^2}\Pi^{\mathrm{ren}}\left(p^2\right)}$$

the evolution is governed by the *renormalized* quantity

$$\Pi^{\mathrm{ren}}\left(p^2\right) = \Pi\left(p^2\right) - \Pi(0)$$

2) the renormalized vertex, $V_1^{\mathrm{ren}}(Q^2;m,m)$, entering the $ee\gamma$ vertex

$$\Lambda_\mu = (2\pi)^4 i \frac{ie^3}{16\pi^2}\left[\gamma_\mu V_1^{\mathrm{ren}}\left(Q^2;m,m\right) + \sigma_{\mu\nu}(p_1+p_2)_\nu V_2\left(Q^2;m,m\right)\right]$$

The *renormalized* vertex is again the difference

$$V_1^{\mathrm{ren}}\left(Q^2;m,m\right) = V_1\left(Q^2;m,m\right) - V_1(0;m,m)$$

Integral representations, limiting cases

$$\begin{aligned}\Pi^{\rm ren}\left(p^2\right) &= \frac{1}{9}+\frac{1}{3}\left(1-2\,\frac{m^2}{p^2}\right)\int_0^1 dx\ln\frac{\chi\left(p^2,x\right)}{m^2}\\ \chi\left(p^2,x\right) &= p^2x\left(1-x\right)+m^2\end{aligned}$$

For low $s=-p^2$

$$\Pi^{\rm ren}\left(p^2\right)=\frac{p^2}{15\,m^2}\,,\qquad \text{for}\quad p^2\to 0$$

the well known contribution to the Uehling effect, i.e. the modification of Coulomb's law due to vacuum polarization.
Alternatively for large $s=-p^2$ we have

$$\Pi^{\rm ren}\left(p^2\right)=\frac{1}{3}\left(\ln\frac{s}{m^2}-i\,\pi\right),\qquad \text{for}\quad s=-p^2\to\infty$$

The $V_1^{\rm ren}\left(Q^2;m,m\right)$ in an integral form

$$\begin{aligned}V_1^{\rm ren}\left(Q^2;m,m\right) &= 2\left(\frac{1}{\hat{\varepsilon}}+\ln\frac{m^2}{\mu^2}\right)\left[1-\frac{Q^2+2m^2}{2}\int_0^1 dx\frac{1}{\chi\left(Q^2,x\right)}\right]\\ &\quad-\left(Q^2+2m^2\right)\int_0^1 dx\frac{1}{\chi\left(Q^2,x\right)}\ln\frac{\chi\left(Q^2,x\right)}{m^2}\\ &\quad-\int_0^1 dx\ln\frac{\chi\left(Q^2,x\right)}{m^2}\\ &\quad+2\left(Q^2+3m^2\right)\int_0^1 dx\frac{1}{\chi\left(Q^2,x\right)}-6\end{aligned}$$

there remains a pole and a scale dependent factor

$$\frac{1}{\hat{\varepsilon}}+\ln\frac{m^2}{\mu^2}$$

which has an infrared origin and which will be compensated in any realistic calculation by the contribution of the real *soft photons* emission and also by the *box* diagrams, which are ultraviolet finite by themselves.

Non-minimal OMS renormalization scheme in the U gauge
Independent quantities of the scheme:
the electric charge, the masses of all particles and all fields

$$\begin{array}{rclcrcl} \psi^i_{0L} &=& \left(Z_L^{1/2}\right)_{ij} \psi^j_L & \qquad & \psi^i_{0R} &=& \left(Z_R^{1/2}\right)_{ij} \psi^j_R \\ W_{0\mu} &=& Z_W^{1/2} W_\mu & & Z_{0\mu} &=& Z_Z^{1/2} Z_\mu \\ H_0 &=& Z_H^{1/2} H & & A_{0\mu} &=& Z_A^{1/2} A_\mu + Z_M^{1/2} Z_\mu \end{array}$$

Bosonic mass renormalization

$$M^2 = Z_{M_W} Z_W^{-1} M_W^2 \,, \quad M_0^2 = Z_{M_Z} Z_Z^{-1} M_Z^2 \,, \quad M_{0H}^2 = Z_{M_H} Z_H^{-1} M_H^2 \quad (*)$$

Fermionic mass renormalization is more involved, due to mixing

$$\mathcal{L}_{\rm ct} \sim -\left(\overline{\psi}_L Z_{m_f} \psi_R + \overline{\psi}_R Z_{m_f}^+ \psi_L - \overline{\psi} m_f \psi\right)$$

All but one renormalization constants are fixed by requiring that the residue of *all* the propagators is 1. This remaining renormalization constant is associated with the renormalization of the electric charge

$$e_0 = Z_e Z_A^{-1/2} e$$

One may use the additive renormalization of the electric charge

$$\begin{array}{rcl} e_0^2 &=& e^2 + \delta e^2 \\ \dfrac{\delta e^2}{e^2} &=& 2\left(Z_e - 1\right) - \left(Z_A - 1\right) \end{array}$$

One may prove that the relevant Ward identity implies

$$Z_e \equiv 1$$

Two definitions, valid to all orders in perturbation theory
The OMS weak mixing angle, θ_W ($c_W = \cos\theta_W$)

$$M_Z^2 c_W^2 = M_W^2$$

The OMS weak charge

$$g^2 = \frac{e^2}{s_W^2} \qquad \left(s_W^2 = 1 - c_W^2 = 1 - \frac{M_W^2}{M_Z^2} \right)$$

The kinetic and mass terms for bosonic fields

$$\begin{aligned}
\mathcal{L}_{\rm ct}^{{\rm kin},A} &= -\frac{1}{4}\left(Z_A-1\right)\left(A_{\mu\nu}\right)^2 \\
\mathcal{L}_{\rm ct}^{{\rm kin},Z} &= -\frac{1}{4}\left(Z_Z+Z_{_M}-1\right)\left(Z_{\mu\nu}\right)^2-\frac{1}{2}\left(Z_{M_Z}-1\right)M_{_Z}^2\left(Z_\mu\right)^2 \\
&\quad -\frac{1}{2}Z_A^{1/2}Z_{_M}^{1/2}A_{\mu\nu}Z_{\mu\nu} \\
\mathcal{L}_{\rm ct}^{{\rm kin},W} &= -\frac{1}{2}\left(Z_W-1\right)|W_{\mu\nu}|^2-\left(Z_{M_Z}-1\right)M_{_W}^2|W_\mu|^2 \\
\mathcal{L}_{\rm ct}^{{\rm kin},H} &= -\frac{1}{2}\left(Z_H-1\right)\left(\partial_\mu H\right)^2-\frac{1}{2}\left(Z_{M_H}-1\right)M_{_H}^2H^2
\end{aligned}$$

Here

$$V_{\mu\nu}=\partial_\mu V_\nu-\partial_\nu V_\mu$$

The fermionic kinetic term

$$\mathcal{L}_{\rm ct}^{{\rm kin},f}=-\frac{1}{2}\overline{\psi}\partial\!\!\!/\left[\left(\sqrt{Z_L}^{\,\dagger}\sqrt{Z_L}-I\right)\gamma_++\left(\sqrt{Z_R}^{\,\dagger}\sqrt{Z_R}-I\right)\gamma_-\right]\psi$$

Since $\sqrt{Z_L}$ and $\sqrt{Z_R}$ are understood as matrices acting in the full fermionic-flavour space, the following equation

$$\left|\sqrt{Z_L}\right|^2=\sqrt{Z_L}^{\,\dagger}\sqrt{Z_L}$$

should be undestood as a notation. In general, these matrices are non-diagonal and even non-hermitian, due to the mixing induced by loop corrections. Renormalization requirement fixes the combinations,

$$\left|\sqrt{Z_L}\right|^2-I,\qquad\left|\sqrt{Z_R}\right|^2-I$$

which directly enters the kinetic term.

In the one loop approximation we may consistently accept that $\sqrt{Z_{L,R}}$ are hermitian matrices, then

$$\sqrt{Z_{L,R}}-I=\frac{1}{2}\left(\left|\sqrt{Z_{L,R}}\right|^2-I\right)$$

and all combinations entering the interaction Lagrangian become known.

The $V(H)f\overline{f}$ interaction parts of the Lagrangian

$$\mathcal{L}_{\rm ct}^{\gamma f\overline{f}} = \frac{i}{2}eQ_f\overline{\psi}\gamma_\mu\Big[\Big(\left|\sqrt{Z_L}\right|^2 - I\Big)\gamma_+ + \Big(\left|\sqrt{Z_R}\right|^2 - I\Big)\gamma_- + 2\left(Z_e - 1\right)\Big]\psi A_\mu$$

$$\begin{aligned}\mathcal{L}_{\rm ct}^{Zf\overline{f}} &= \frac{i}{2}\frac{e}{s_W c_W}\overline{\psi}\gamma_\mu\left\{\left[\left|\sqrt{Z_L}\right|^2\left(\frac{Z_{M_Z}Z_W}{Z_A Z_{M_W}Z_c}\right)^{1/2} - I\right] I_f^{(3)}\gamma_+\right. \\ &\quad -2Q_f s_W^2\left[\frac{1}{2}\left(\left|\sqrt{Z_L}\right|^2\gamma_+ + \left|\sqrt{Z_R}\right|^2\gamma_-\right)\left(\frac{Z_{M_Z}Z_W}{Z_A Z_{M_W}Z_c}\right)^{1/2} - I\right] \\ &\quad \left. -2Q_f s_W c_W\left(\frac{1}{2}\left|\sqrt{Z_L}\right|^2\gamma_+ + \frac{1}{2}\left|\sqrt{Z_R}\right|^2\gamma_-\right)\left(\frac{Z_M}{Z_A}\right)^{1/2}\right\}\psi Z_\mu \\ \mathcal{L}_{\rm ct}^{Wf\overline{f}'} &= \frac{i}{2\sqrt{2}}\frac{e}{s_W}\overline{\psi}^u\gamma_\mu\gamma_+\left[\sqrt{Z_{uL}}^{\dagger}C\sqrt{Z_{dL}}\left(\frac{Z_W}{Z_A Z_c}\right)^{1/2} - C\right]\psi^d + h.c. \\ \mathcal{L}_{\rm ct}^{Hf\overline{f}} &= -\frac{e}{2M_W s_W}\overline{\psi}\left[\frac{1}{2}\left(Z_{m_f}\gamma_- + Z_{m_f}^{+}\gamma_+\right)\left(\frac{Z_H Z_W}{Z_A Z_{M_W}Z_c}\right)^{1/2} - m_f\right]\psi\end{aligned}$$

where

$$Z_c = 1 - \frac{\delta c_W^2}{s_W^2}\,, \qquad s_W^2 = 1 - \frac{M_W^2}{M_Z^2}\,, \qquad \frac{\delta c_W^2}{c_W^2} = \frac{\delta M_W^2}{M_W^2} - \frac{\delta M_Z^2}{M_Z^2}$$

with M_W and M_Z being the physical masses of the vector bosons, C being the CKM-mixing matrix

Full list of bosonic renormalization constants (some unnatural looking is an artifact of the definition (*))

$$\begin{aligned}Z_{M_W} - Z_W &= \frac{\delta M_W^2}{M_W^2} = \frac{g^2}{16\pi^2 M_W^2}\Sigma_{WW}\left(M_W^2\right) \\ Z_{M_Z} - Z_Z &= \frac{\delta M_Z^2}{M_Z^2} = \frac{g^2}{16\pi^2 c_\theta^2 M_Z^2}\Sigma_{ZZ}\left(M_Z^2\right) \\ Z_{M_H} - Z_H &= \frac{\delta M_H^2}{M_H^2} = \frac{g^2}{16\pi^2 M_H^2}\Sigma_{HH}\left(M_H^2\right) \\ Z_M^{1/2} &= \frac{g^2 s_W}{16\pi^2 c_W M_Z^2}\Sigma_{ZA}\left(M_Z^2\right)\end{aligned}$$

$$
\begin{aligned}
Z_A - 1 &= \frac{e^2}{16\pi^2}\Pi_{\gamma\gamma}(0) \\
Z_Z - 1 &= \frac{g^2}{16\pi^2 c_\theta^2}\frac{\partial\Sigma_{ZZ}(p^2)}{\partial p^2}\Big|_{p^2=-M_Z^2} \\
Z_W - 1 &= \frac{g^2}{16\pi^2}\frac{\partial\Sigma_{WW}(p^2)}{\partial p^2}\Big|_{p^2=-M_W^2} \\
Z_H - 1 &= \frac{g^2}{16\pi^2}\frac{\partial\Sigma_{HH}(p^2)}{\partial p^2}\Big|_{p^2=-M_H^2}
\end{aligned}
$$

Argument convention.

For every: Σ_{VV}, $\Pi_{\gamma\gamma}$, Π_{3Q}..., if $p^2 = -s$ or $p^2 = -M^2$, then we will omit the minus sign, i.e. $\Sigma_{VV}(s)$... On the contrary, in the argument list of B_k, C_k... functions, we will maintain explicitly the sign.

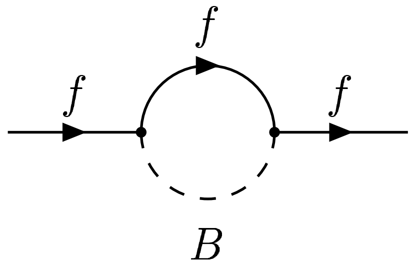
Linearization.

Due to perturbation theory, where all renormalization constants are power series in the coupling constant e^2.

$$
\begin{aligned}
\mathcal{L}_{\rm ct}^{Zf\overline{f}} &= \frac{i}{2}\frac{e}{s_W c_W}\overline{\psi}\gamma_\mu\Big\{\Big[\left|\sqrt{Z_L}\right|^2 - I \\
&+\frac{1}{2}\left((Z_Z-1)-(Z_A-1)+\frac{c_W^2-s_W^2}{s_W^2}\frac{\delta c_W^2}{c_W^2}\right)\Big] I_f^{(3)}\gamma_+ \\
&-2Q_f s_W^2\left[\frac{1}{2}\left(\left|\sqrt{Z_L}\right|^2 - I\right)\gamma_+ + \frac{1}{2}\left(\left|\sqrt{Z_R}\right|^2 - I\right)\gamma_-\right. \\
&\left.+\frac{1}{2}\left((Z_Z-1)-(Z_A-1)-\frac{1}{s_W^2}\frac{\delta c_W^2}{c_W^2}\right)+\frac{c_W}{s_W}Z_M^{1/2}\right]\Big\}\psi Z_\mu \\
\mathcal{L}_{\rm ct}^{Wf\overline{f}'} &= \frac{i}{2\sqrt{2}}\frac{e}{s_W}\overline{\psi}^u\gamma_\mu\gamma_+\Big\{\left(\sqrt{Z_{uL}} - I\right)C + C\left(\sqrt{Z_{dL}} - I\right) \\
&+C\left[\frac{1}{2}(Z_W-1)-\frac{1}{2}(Z_A-1)+\frac{\delta c_W^2}{2s_W^2}\right]\Big\}\psi^d + h.c. \\
\mathcal{L}_{\rm ct}^{Hf\overline{f}} &= -\frac{e}{2M_W s_W}\overline{\psi}\Big\{\left(Z_{m_f}-m_f\right)+m_f\Big[\frac{1}{2}(Z_H-1)-\frac{1}{2}\left(Z_{M_W}-1\right) \\
&+\frac{1}{2}(Z_W-1)-\frac{1}{2}(Z_A-1)+\frac{1}{2}\frac{\delta c_W^2}{s_W^2}\Big]\Big\}\psi
\end{aligned}
$$

Fermionic renormalization constants

Fermionic self-energy



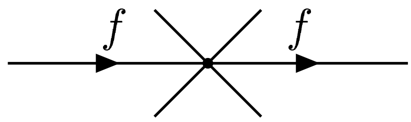
The most general expression

$$\Sigma(i\not{p}) = (2\pi)^4\, i \left[a_1 + a_2\,\gamma_5 + (a_3 - a_4\,\gamma_5)\, i\not{p}\right]$$

In the Standard Model always $a_2 = 0$

$$\Sigma(i\not{p}) = (2\pi)^4\, i \left[a_1 + a_3 i\not{p} + a_4 i\not{p}\gamma_5\right]$$

Kinetic and mass terms of the Lagrangian may be depicted as



and they contribute as

$$\begin{aligned} &-\frac{1}{2} i\not{p}\left[\left(\left|\sqrt{Z_L}\right|^2 - I\right)\gamma_+ + \left(\left|\sqrt{Z_R}\right|^2 - I\right)\gamma_-\right] \\ &-\left(Z_{m_f} - m_f\right) \end{aligned}$$

From requirement that the sum vanishes on fermion mass shell, one derives all renormalization constants

$$\begin{aligned} \left|\sqrt{Z_L}\right|^2 - I &= a_3 - 2m^2 a_3' + 2m a_1' + a_4 \\ \left|\sqrt{Z_R}\right|^2 - I &= a_3 - 2m^2 a_3' + 2m a_1' - a_4 \\ Z_{m_f} &= \ldots \end{aligned}$$

note derivatives $a_i' = \partial a_i/\partial p^2|_{p^2=-m^2}$

Summary of Level 4

1) Standard Model, Fields and Lagrangian
   Feynman Rules → *building* of diagrams

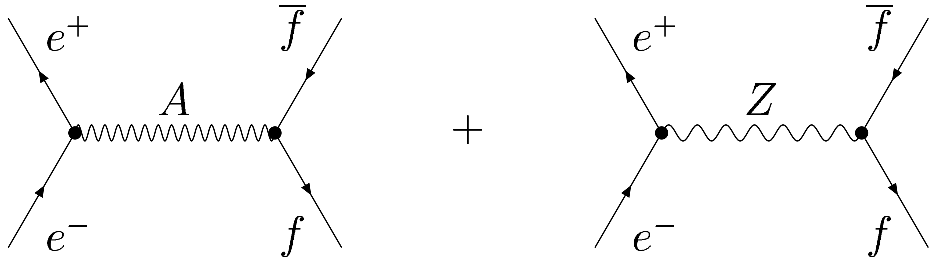
2) Regularization:
   N-point functions → *calculation* of diagrams

3) *Building blocks*: Self-energies, Vertices,
   Boxes (direct/crossed)

4) Renormalization:
   Dyson resummation
   Renormalization constants
   Counterterm Lagrangian
   OMS scheme and fixation of renormalization constants
   Residue One requirement

5) Time to calculate amplitudes of physical processes

The Born amplitude and diagrams for the reaction $e^+e^- \to f\overline{f}$



$$\mathcal{A}_\gamma^{\rm Born} = \frac{e^2 Q_e Q_f}{s} \gamma_\mu \otimes \gamma_\mu$$

$$\mathcal{A}_Z^{\rm Born} = \frac{e^2}{4 s_W^2 c_W^2} \chi_Z(s)\, \gamma_\mu (v_e + a_e \gamma_5) \;\otimes\; \gamma_\mu (v_f + a_f \gamma_5) \quad - \quad \text{VA-basis}$$

$$\mathcal{A}_Z^{\rm Born} = \frac{e^2}{4 s_W^2 c_W^2} \chi_Z(s)\, \gamma_\mu \left[I_e^{(3)} \gamma_+ - 2 Q_e s_W^2 \right] \otimes \gamma_\mu \left[I_f^{(3)} \gamma_+ - 2 Q_f s_W^2 \right] \quad - \quad \text{LQ-basis}$$

where $\gamma_\pm = 1 \pm \gamma_5$ and

$$\chi_Z(s) = \frac{1}{s - M_Z^2 + i s \Gamma_Z / M_Z}$$

From

$$\frac{g^2}{8 M_W^2} = \frac{G_F}{\sqrt{2}}\,, \qquad s_W^2 = \frac{e^2}{g^2}\,, \qquad c_W^2 = \frac{M_W^2}{M_Z^2}$$

one easily derives

$$\frac{e^2}{4 s_W^2 c_W^2} = \sqrt{2} G_F M_Z^2$$

or, using $e^2 = 4\pi\alpha$, define *conversion factor*

$$f = \frac{\sqrt{2} G_F M_Z^2 s_W^2 c_W^2}{\pi \alpha} \quad (\neq 1 \ \text{due to loop corrections})$$

Muon decay

Muon lifetime. The process is

$$\mu \to e + \nu_\mu + \overline{\nu}_e$$

If QED corrections and W-propagator effects are included (PDG)

$$\tau_\mu^{-1} = \frac{G_F^2 m_\mu^5}{192\,\pi^3} F\left(\frac{m_e^2}{m_\mu^2}\right)\left(1+\frac{3}{5}\frac{m_\mu^2}{M_W^2}\right)\left[1+\frac{\alpha\left(m_\mu^2\right)}{2\,\pi}\left(\frac{25}{4}-\pi^2\right)\right]$$

$$F(r) = 1 - 8\,r + 8\,r^3 - r^4 - 12\,r^2 \ln r\,, \qquad \alpha^{-1}\left(m_\mu^2\right) \approx 136$$

One must use the accurately measured muon lifetime or the Fermi coupling constant, $G_F = 1.16639(2) \times 10^{-5}\ \mathrm{GeV}^{-2}$.

This decay process is conventionally described with the effective four-fermion Fermi Lagrangian

$$\mathcal{L}_F = \frac{G_F}{\sqrt{2}}\overline{\psi}_e \gamma_\mu \gamma_+ \psi_\mu \overline{\psi}_{\nu_\mu} \gamma_\mu \gamma_+ \psi_{\nu_e} + \mathrm{h.c.}$$

One calculates observable distribution, $dI^0(x)$, in kinematical variable

$$x = \frac{2E_e}{m_\mu}\,, \qquad E_e \text{ is the electron energy}$$

If electron mass is neglected, x varies from 0 to 1. In lowest order

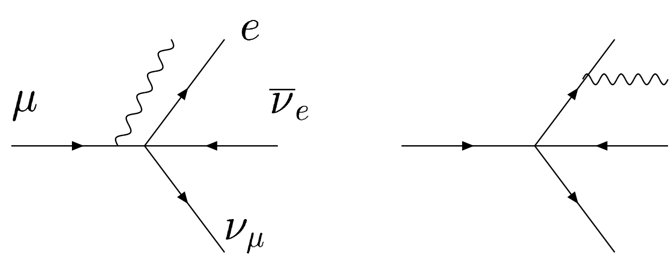
$$dI^0(x) = \frac{G_F^2 m_\mu^5}{96\,\pi^3} x^2\left(3 - 2\,x\right) \;\to\; \frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\,\pi^3}$$

Real and virtual QED corrections

Bremsstrahlung in μ-decay

$$\mu \to e + \nu_\mu + \overline{\nu}_e + \gamma$$

It is described by diagrams

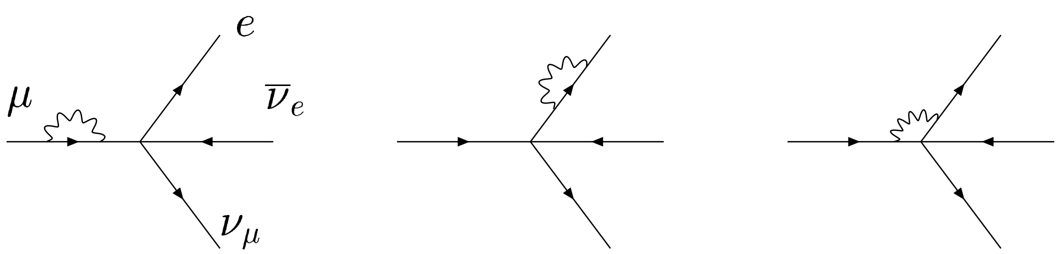


The quantity of experimental interest is the transition probability summed over all possible photons in the final state. After lenghy calculations

$$
\begin{aligned}
dI^{\mathrm{r}}(x) &= \frac{G_F^2 m_\mu^5}{96\,\pi^3}\,\frac{d\Omega_e}{4\,\pi}\,\frac{\alpha}{2\,\pi}\,\mathcal{I}(x) \\
\mathcal{I}(x) &= 2\,x^2\,(2\,x-3)\,\left(L-1\right)\left[\frac{1}{\hat{\varepsilon}}+\ln\frac{m_\mu m_e}{\mu^2}+\ln\frac{(1-x)^2}{x}\right] \\
&\quad +2\,x\left(3-5\,x+2\,x^2\right)\,\ln\left(1-x\right) \\
&\quad +\left(-\frac{5}{3}-4\,x+17\,x^2-\frac{34}{3}\,x^3\right)\left(L-1\right) \\
&\quad +2\,x^2\,(2\,x-3)\,\mathrm{Li}_2\left(x\right)+x^2\,\left(1-\frac{2}{3}\,x\right)\,\pi^2-\frac{5}{3}\,\left(1-x\right)^2 \\
L &= \ln\left(x\frac{m_\mu}{m_e}\right)
\end{aligned}
$$

<u>Virtual QED corrections for μ-decay</u>

There are three diagrams contributing to the order α



Lowest order interaction $\bar{u}_e\gamma_\alpha\gamma_+ u_\mu$ receives QED corrections (dressing)

$$
-\frac{\alpha}{4\,\pi}\,\left(\mathcal{F}_1\gamma_\alpha\gamma_+ +\frac{i}{m_\mu}F_2\,q_{\mu,\alpha}\gamma_- +\frac{i}{m_\mu}F_3\,q_{e,\alpha}\gamma_-\right)
$$

Result of calculation of diagrams:

$$
\begin{aligned}
\mathcal{F}_1 &= 2\left(\frac{1}{\hat{\varepsilon}}+\ln\frac{m_\mu m_e}{\mu^2}\right)\left(L-1\right)+2\,\zeta(2)-2\,\mathrm{Li}_2\left(x\right) \\
&\quad +\ln x\,\left[\frac{1}{1-x}-2\ln\left(1-x\right)+2L\right]-3L+4 \\
F_2 &= \frac{2}{\left(1-x\right)^2}\,\left[1-x+x\,\ln x\right] \\
F_3 &= -\frac{2}{\left(1-x\right)^2}\,\left[1-x+\left(2\,x-1\right)\,\ln x\right]
\end{aligned}
$$

After calculation of traces

$$dI^{0+\mathrm{v}}(x) = \left(1 - \frac{\alpha}{2\,\pi}\mathcal{F}_1\right)\; I^{(0)}(x) + \frac{G_F^2 m_\mu^5}{96\,\pi^3}\,\frac{\alpha}{4\,\pi}\,x^3\;(F_2 + F_3)$$

The lowest order result is multiplied by a correction factor, $\mathcal{F}_1$, which is ultraviolet but not infrared finite; the remaining two form factors, F_2 and F_3, are finite (the latter are *induced* form factors). This expression is infrared divergent and must be combined with $dI^{\mathrm{r}}(x)$.

Total QED corrections for μ-decay

The experimentally observable quantity is the sum of the two transition probabilities for real and virtual processes.

$$\begin{aligned}
dI(x) &= \frac{G_F^2 m_\mu^5}{96\,\pi^3}\,x^2\,\left[3 - 2\,x + \frac{\alpha}{2\,\pi}\Delta I(x)\right] \\
\Delta I(x) &= 2\;(3 - 2\,x)\;\left[(L-1)\;\left(2\;\ln\frac{1-x}{x} + \frac{3}{2}\right)\right. \\
&\quad \left. + \ln(1-x)\;\left(\ln x + 1 - \frac{1}{x}\right) - \ln x + 2\,\mathrm{Li}_2(x) - \frac{1}{3}\,\pi^2 - \frac{1}{2}\right] \\
&\quad -3\;\ln x + \frac{1-x}{3\,x^2}\;\left[\left(5 + 17\,x - 34\,x^2\right)\;L - 22\,x - 34\,x^2\right]
\end{aligned}$$

By integrating $dI(x)$ over x from 0 to 1

$$\frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\,\pi^3}\,\left[1 + \frac{\alpha}{2\,\pi}\;\left(\frac{25}{4} - \pi^2\right)\right]$$

This the result derived within QED $\otimes$ effective 4-fermion Fermi theory.

Of course, calculation could be performed exclusively within the Standard Model framework. This would give something like

$$\frac{1}{\tau_\mu} = \frac{m_\mu^5}{192\pi^3}\,\frac{g^4}{32\,M_W^4}\;(1 + \delta_\mu)\,.$$

However, Fermi constant was historically *defined* by equation (*).

The important point is that both the EW and QED corrections are infrared and ultraviolet finite and gauge invariant, therefore they can be treated separately.

EW corrections for muon decay

As a consequence

$$\delta_\mu = \delta_\mu^{em} + \delta_\mu^{ew}$$

or

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}\left[1+\frac{1}{2}\left(\delta_\mu - \delta_\mu^{em}\right)\right] = \frac{g^2}{8M_W^2}\left(1+\frac{\alpha}{4\pi}\delta_\mu^{ew}\right)$$

remember basic definitions of OMS scheme

$$M_Z^2 c_W^2 = M_W^2\,, \qquad g^2 = \frac{e^2}{s_W^2}$$

using them, we arrive at

$$s_W^2 c_W^2 = \frac{\pi\alpha}{\sqrt{2}G_F M_Z^2}\left(1+\Delta r\right), \qquad \Delta r = \frac{\alpha}{4\pi}\delta_\mu^{ew}$$

or (Sirlin, 1980)

$$s_W^2 c_W^2 = \frac{\pi\alpha}{\sqrt{2}G_F M_Z^2}\frac{1}{1-\Delta r}$$

the final result for Δr in one loop approximation

$$\begin{aligned}\Delta r \;=\; & \frac{\alpha}{4\pi}\frac{1}{s_W^2}\left\{s_W^2\left[-\frac{2}{3}-\Pi_{\gamma\gamma}^{\text{fer},F}(0)\right]+\frac{c_W^2}{s_W^2}\Delta\rho^F\right.\\ & \left.+\Delta\rho_W^F+\frac{11}{2}-\frac{5}{8}c_W^2\left(1+c_W^2\right)+\frac{9c_W^2}{4s_W^2}\ln c_W^2\right\}\end{aligned}$$

where *Finite* parts of $\Delta\rho^F$ factors

$$\Delta\rho^F = \Delta\rho^{\text{bos},F}+\Delta\rho^{\text{fer},F}, \qquad \Delta\rho_W^F = \Delta\rho_W^{\text{bos},F}+\Delta\rho_W^{\text{fer},F}$$

have *fer*mionic and *bos*onic contributions

$$\begin{aligned}\Delta\rho^{\text{bos(fer)},F} &= \frac{1}{M_W^2}\left[\Sigma_{WW}^{\text{bos(fer)},F}\left(M_W^2\right)-\Sigma_{ZZ}^{\text{bos(fer)},F}\left(M_Z^2\right)\right]\\ \Delta\rho_W^{\text{bos(fer)},F} &= \frac{1}{M_W^2}\left[\Sigma_{WW}^{\text{bos(fer)},F}(0)-\Sigma_{WW}^{\text{bos(fer)},F}\left(M_W^2\right)\right]\end{aligned}$$

bosonic contributions explicitely

$$\begin{aligned}
\Delta\rho_W^{\mathrm{bos},F} &= -\left(\frac{1}{12c_W^4}+\frac{4}{3c_W^2}-\frac{17}{3}-4c_W^2\right)B_0^F\left(-M_W^2;M_Z,M_W\right)\\
&-\left(1-\frac{1}{3}w_h+\frac{1}{12}w_h^2\right)B_0^F\left(-M_W^2;M_H,M_W\right)\\
&+\left[\frac{3}{4\left(1-w_h\right)}+\frac{1}{4}-\frac{1}{12}w_h\right]w_h\ln w_h\\
&+\left(\frac{1}{12c_W^4}+\frac{17}{12c_W^2}-\frac{3}{s_W^2}+\frac{1}{4}\right)\ln c_W^2\\
&+\frac{1}{12c_W^4}+\frac{11}{8c_W^2}+\frac{139}{36}-\frac{177}{24}c_W^2+\frac{5}{8}c_W^4-\frac{1}{12}w_h\left(\frac{7}{2}-w_h\right)\\
\Delta\rho^{\mathrm{bos},F} &= -\left(\frac{1}{12c_W^2}+\frac{4}{3}-\frac{17}{3}c_W^2-4c_W^4\right)\\
&\times\left[B_0^F\left(-M_Z^2;M_W,M_W\right)-\frac{1}{c_W^2}B_0^F\left(-M_W^2;M_Z,M_W\right)\right]\\
&+\left(1-\frac{1}{3}w_h+\frac{1}{12}w_h^2\right)B_0^F\left(-M_W^2;M_H,M_W\right)\\
&-\left(1-\frac{1}{3}z_h+\frac{1}{12}z_h^2\right)\frac{1}{c_W^2}B_0^F\left(-M_Z^2;M_H,M_Z\right)\\
&+\frac{1}{12}s_W^2w_h^2(\ln w_h-1)-\left(\frac{1}{12c_W^4}+\frac{1}{2c_W^2}-2+\frac{1}{12}w_h\right)\ln c_W^2\\
&-\frac{1}{12c_W^4}-\frac{19}{36c_W^2}-\frac{133}{18}+8c_W^2
\end{aligned}$$

where two ratios

$$w_h=\frac{M_H^2}{M_W^2}\,,\qquad z_h=\frac{M_H^2}{M_Z^2}$$

and the finite parts of the B_0 function are introduced

$$B_0\left(p^2;M_1,M_2\right)=\frac{1}{\bar{\varepsilon}}-\ln\frac{M_W^2}{\mu^2}+B_0^F\left(p^2;M_1,M_2\right)$$

Resummation of large corrections. In order to get high precision of theoretical predictions, one has to improve upon the one loop expression. We begin with the extraction of $\Delta\alpha^{\text{fer}}\left(M_Z^2\right)$ from Δr. From the definition of $\Delta\alpha^{\text{fer}}\left(M_Z^2\right)$,

$$\alpha^{\text{fer}}\left(M_Z^2\right) = \frac{\alpha}{1-\Delta\alpha^{\text{fer}}\left(M_Z^2\right)}$$

and the definition of the e.m. running coupling

$$\alpha(s) = \frac{\alpha}{1-\frac{\alpha}{4\pi}\Pi^F(s)}, \qquad \text{with} \qquad \Pi^F(s) = \Pi_{\gamma\gamma}(s) - \Pi_{\gamma\gamma}(0)$$

we derive the following representation for Δr

$$\begin{aligned}\Delta r &= \Delta\alpha^{\text{fer}}\left(M_Z^2\right) + \frac{\alpha}{4\pi s_W^2}\left\{s_W^2\left[-\frac{2}{3} - \Pi_{\gamma\gamma}^{t,F}(0) - \Pi_{\gamma\gamma}^{l+5q,F}\left(M_Z^2\right)\right]\right. \\ &\left. + \frac{c_W^2}{s_W^2}\Delta\rho^F + \Delta\rho_W^F + \frac{11}{2} - \frac{5}{8}c_W^2\left(1+c_W^2\right) + \frac{9c_W^2}{4s_W^2}\ln c_W^2\right\}\end{aligned}$$

where the superscript $l+5q$ stands for a summation over leptons and five light quarks.

NB: The $\Delta\alpha^{\text{fer}}\left(M_Z^2\right)$ is defined at the scale $\mu = M_Z$ $\rightarrow$ rescale relevant quantities to the *natural* value $\mu = M_Z$. The quantity $\Delta\rho^F$, evaluated at $\mu = M_Z$, is a gauge invariant object and therefore a good candidate for re-summation. Define the leading and remainder contributions to Δr

$$\begin{aligned}\Delta r_L &= -\frac{\alpha}{4\pi}\frac{c_W^2}{s_W^4}\Delta\rho^F\Big|_{\mu=M_Z} \\ \Delta r_{\text{rem}} &= \frac{\alpha}{4\pi s_W^2}\left\{s_W^2\left[-\frac{2}{3} - \Pi_{\gamma\gamma}^{t,F}(0) - \Pi_{\gamma\gamma}^{l+5q,F}\left(M_Z^2\right)\right]\right. \\ &+ \left(\frac{1}{6}N_f - \frac{1}{6} - 7c_W^2\right)\ln c_W^2 \\ &\left. + \Delta\rho_W^F + \frac{11}{2} - \frac{5}{8}c_W^2\left(1+c_W^2\right) + \frac{9c_W^2}{4s_W^2}\ln c_W^2\right\}\Big|_{\mu=M_Z}\end{aligned}$$

The re-summed one-loop representation

$$\frac{\sqrt{2}G_F M_Z^2 s_W^2 c_W^2}{\pi\alpha} = \frac{1}{\left(1 - \Delta\alpha^{\text{fer}}\left(M_Z^2\right) - \Delta r_{\text{rem}}\right)\left(1 + \frac{\sqrt{2}G_F M_Z^2 s_W^2 c_W^2}{\pi\alpha}\Delta r_L\right)} \qquad (*)$$

The re-summation of $\Delta\alpha^{\text{fer}}\left(M_Z^2\right)$ is dictated by renormalization group arguments. The re-summation of terms containing $\Delta\rho^{\text{fer},F}$ (Δr_L) finds its roots in the two loop EW calculation. The equation (*) is therefore an improved upon the one-loop result.

Higher orders, in particular QCD corrections of the order $\mathcal{O}\left(\alpha\alpha_S\right)$ and second order electroweak corrections $\mathcal{O}\left(G_F^2 m_t^4\right)$ and $\mathcal{O}\left(G_F^2 m_t^2 M_Z^2\right)$ are applied by means of modifications of the leading and remainder terms.

$$\Delta r_L \to \Delta r_L + \Delta r_L^{\text{ho}}, \qquad \Delta r_{\text{rem}} \to \Delta r_{\text{rem}} + \Delta r_{\text{rem}}^{\text{ho}} .$$

An iterative solution of (*) for the M_W which incorporates second order electroweak corrections, without and with QCD correction is shown in the Table.

m_t [GeV]	M_H [GeV]		
	65	300	1000
170.1	80.445	80.349	80.256
	80.375	80.279	80.186
175.6	80.482	80.386	80.291
	80.409	80.312	80.219
181.1	80.521	80.423	80.329
	80.444	80.346	80.252

Table 1: *The W-boson mass, M_W [GeV] in OMS scheme, $\alpha_S = 0$ — first entry, $\alpha_S = 0.120$ — second entry.*

Z Resonance Observables at one loop

Definition 1 *Realistic Observables. They are the cross-sections $\sigma^f(s)$ and asymmetries $A^f(s)$ of the reactions*

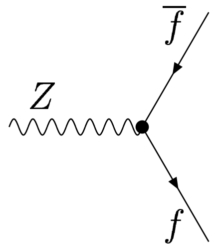
$$e^+e^- \to (\gamma, Z) \to f\overline{f}(n\gamma)$$

calculated for a given value of $s = 4E^2$ with all available higher order corrections (QCD, EW), including real and virtual QED photonic corrections, possibly accounting for kinematical cuts.

Definition 2 *Pseudo-Observables. They are related to measured cross-sections and asymmetries by some de-convolution or unfolding procedure (i.e. undressing of QED corrections). The concept itself of pseudo-observability is rather difficult to define. One way to introduce it is to say that the experiments measure some primordial (basically cross-sections and thereby asymmetries also) quantities which are then reduced to secondary quantities under some set of specific assumptions. Within these assumptions, the secondary quantities, the pseudo-observables, also deserve the label of observability.*

The Z partial widths

Born diagram and amplitudes



In the VA-basis,

$$V_\mu^{Zf\overline{f}} = (2\pi)^4 \, i \, \frac{ig^3}{16\,\pi^2 c_W} \gamma_\mu \left[v_f + a_f \, \gamma_5 \right]$$

In the LQ-basis,

$$V_\mu^{Zf\overline{f}} = (2\pi)^4 \, i \, \frac{ig^3}{16\,\pi^2 c_W} \gamma_\mu \left[I_f^{(3)} \gamma_+ - 2Q_f s_W^2 \right]$$

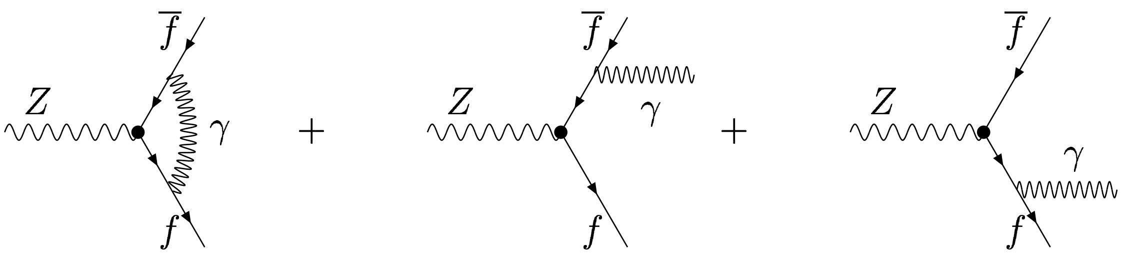
The partial width of $Z \to f\overline{f}$ decay in the Born approximation

$$\Gamma_f^{(0)} = \frac{G_F M_Z^3}{6\sqrt{2}\pi} c_f \left[v_f^2 + a_f^2 \right]$$

with couplings:

$$v_f = I_f^{(3)} - 2Q_f s_W^2 \,, \qquad a_f = I_f^{(3)}$$

QED diagrams and corrections



QED diagrams are separately gauge invariant and finite. Their contibution integrated over full bremsstrahlung photon phase space is

$$\Gamma_f^{\text{QED}} = \Gamma_f^{(0)} \left(1 + \frac{3\alpha}{4\pi} Q_f^2 \right)$$

The $Z \to f\overline{f}$ decay amplitude in OMS scheme

One loop diagrams and amplitutes

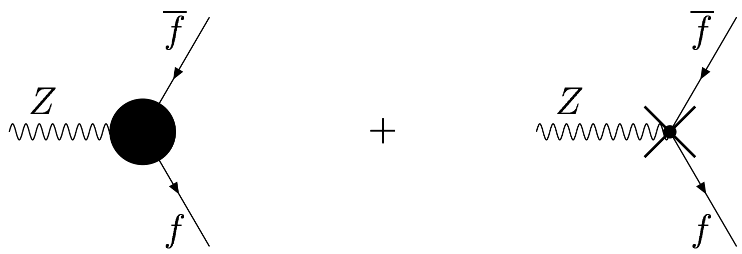


Figure 13: Process $Z \to f\overline{f}$; fermion vertex and its counterterms

In the VA-basis,

$$V_\mu^{Zf\overline{f}} = (2\pi)^4\, i\, \frac{ig^3}{16\,\pi^2 c_W}\, \gamma_\mu \left[F_V(M_Z^2) + F_A(M_Z^2)\,\gamma_5\right]$$

In the LQ-basis,

$$V_\mu^{Zf\overline{f}} = (2\pi)^4\, i\, \frac{ig^3}{16\,\pi^2 c_W}\, \gamma_\mu \left[I_f^{(3)} F_L\left(M_Z^2\right)\gamma_+ - 2Q_f s_W^2 F_Q\left(M_Z^2\right)\right]$$

Out of this amplitude one constructs the Z partial widths, Γ_f, which can be compared with the experimental data.

The Z width in one loop approximation.

Consider the sum of the Born and of the one loop corrected amplitudes for the Z boson decay.

$$\begin{aligned} V_\mu^{\rm cor}\left(M_Z^2\right) &\propto \frac{ie}{2s_W c_W}\,\gamma_\mu \left[I_f^{(3)} f_{Z,L}\gamma_+ - 2Q_f s_W^2 f_{Z,Q}\right] \\ &= \frac{ie f_{Z,L}}{2s_W c_W}\gamma_\mu \Big[I_f^{(3)}\gamma_+ - 2Q_f s_W^2 (1 + f_{Z,Q} - f_{Z,L})\Big] \end{aligned}$$

where

$$f_{Z,L(Q)} = 1 + \frac{\alpha}{4\pi s_W^2} F_{Z,L(Q)}\left(M_Z^2\right)$$

Using definition of Δr, rewritten as follows,

$$\frac{e}{s_W c_W} = \frac{G_F M_Z^2}{\sqrt{2\sqrt{2}}} \left(1 - \frac{1}{2}\Delta r\right)$$

we eliminate the ratio $e/(s_W c_W)$ in favour of the Fermi constant G_F and $F_{Z,L}\left(M_Z^2\right)$ receives a shift of $-\Delta r/2$. Define the two *effective couplings* ρ_Z^f and κ_Z^f,

$$\begin{aligned}
\rho_Z^f &= 1 + \frac{\alpha}{4\pi s_W^2}\left[2F_{Z,L}\left(M_Z^2\right) - s_W^2 \delta_W\right] \\
\kappa_Z^f &= 1 + \frac{\alpha}{4\pi s_W^2}\left[F_{Z,Q}\left(M_Z^2\right) - F_{Z,L}\left(M_Z^2\right)\right]
\end{aligned}$$

The one loop improved expression for the partial width of $Z \to f\overline{f}$ decay in the OMS scheme

$$\Gamma_f = \frac{G_F M_Z^3}{6\sqrt{2}\pi} c_f \rho_Z^f \left[\left(v_{\text{eff}}^f\right)^2 R_V^f + \left(I_f^{(3)}\right)^2 R_A^f\right]$$

with effective couplings: ρ_Z^f and

$$\begin{aligned}
v_{\text{eff}}^f &= I_f^{(3)} - 2Q_f \sin^2\theta_{\text{eff}}^f \\
\sin^2\theta_{\text{eff}}^f &= \kappa_Z^f s_W^2
\end{aligned}$$

Factors R_V^f and R_A^f accumulate final state QED and QCD corrections. The lowest order QED + QCD result

$$R_V^f = R_A^f = 1 + \frac{3\alpha}{4\pi} Q_f^2 + \frac{\alpha_S}{\pi}$$

By now, more terms are computed and really needed

$$R_V^f = 1 + \frac{3\alpha(M_Z^2)}{4\pi} Q_f^2 + \frac{\alpha_S(M_Z^2)}{\pi} - \frac{\alpha(M_Z^2)}{4\pi} Q_f^2 \frac{\alpha_S(M_Z^2)}{\pi} + C_V^{(2)} \left(\frac{\alpha_S(M_Z^2)}{\pi}\right)^2 + .$$

$$R_A^f = 1 + \frac{3\alpha(M_Z^2)}{4\pi} Q_f^2 + \frac{\alpha_S(M_Z^2)}{\pi} - \frac{\alpha(M_Z^2)}{4\pi} Q_f^2 \frac{\alpha_S(M_Z^2)}{\pi} + C_A^{(2)} \left(\frac{\alpha_S(M_Z^2)}{\pi}\right)^2 + .$$

Re-summation of large corrections in OMS.

Define *leading* (enchanced) and *remainder* contribution to ρ^f_Z and κ^f_Z, similarly to what was done for Δr

$$
\begin{aligned}
\rho^f_Z &= 1 + \rho^f_L + \rho^f_{\rm rem} \\
\kappa^f_Z &= 1 + \kappa^f_L + \kappa^f_{\rm rem}
\end{aligned}
$$

When we eliminate Δr and normalize amplitudes to the Fermi constant G_F, all large corrections containing $\alpha^{\rm fer}\left(M^2_Z\right)$ are automatically accounted for.

Therefore, to the contrary of what happened in the re-summation of Δr, here one has to re-sum only m^2_t-enhanced terms . Similarly to Δr

$$
\rho^f_Z = \frac{1+\rho^f_{\rm rem}}{1+\dfrac{\sqrt{2}G_F M^2_Z s^2_W c^2_W}{\pi\alpha}\rho^f_L}
$$

For κ one has to follow a slightly different procedure

$$
\kappa^f_Z = \left(1+\kappa^f_{\rm rem}\right)\left(1+\frac{\sqrt{2}G_F M^2_Z s^2_W c^2_W}{\pi\alpha}\kappa^f_L\right) + \frac{1}{s^2_W}\text{Im-parts}
$$

where Im-parts are enchanced by $\pi^2 n_f$ second order terms.

The leading contributions are

$$
\rho^f_L = -\frac{\alpha}{4\pi}\frac{1}{s^2_W}\Delta\rho^F, \qquad \kappa^f_L = -\frac{\alpha}{4\pi}\frac{c^2_W}{s^4_W}\Delta\rho^F = \Delta r_L
$$

Inclusion of higher order irreducible effects, are implemented by a modification of the leading and of the reminder terms. Similarly to Δr,

$$
\begin{aligned}
\Delta r_L &\to \Delta r_L + \Delta r^{\rm ho}_L \\
\rho^f_{\rm rem} &\to \rho^f_{\rm rem} + \rho^{f,{\rm ho}}_{\rm rem} \\
\kappa^f_{\rm rem} &\to \kappa^f_{\rm rem} + \kappa^{f,{\rm ho}}_{\rm rem}
\end{aligned}
$$

Numerical results for $\sin^2\theta^e_{\rm eff}$, are derived with inclusion of re-summation of leading corrections and of the leading and sub-leading two loop electroweak corrections $\mathcal{O}\left(G^2_F m^4_t\right)$ and $\mathcal{O}\left(G^2_F m^2_t M^2_Z\right)$.

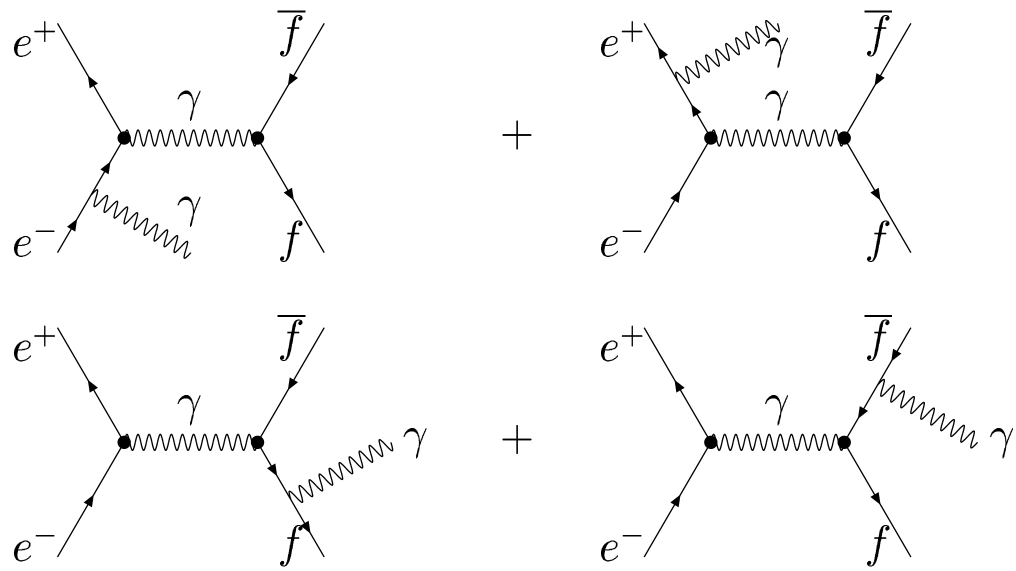
m_t [GeV]	M_H [GeV]		
	65	300	1000
170.1	0.23109	0.23187	0.23253
175.6	0.23090	0.23168	0.23234
181.1	0.23070	0.23149	0.23215

Table 2: *The OMS* $\sin^2\theta^e_{\rm eff}$.

The process $e^+e^- \to f\overline{f}$

For this process, the one loop QED diargams: QED vertices and $\gamma\gamma$ and $Z\gamma$ boxes, form a gauge invariant subset of all diagrams.

It has to be considered together with QED bremsstrahlung diagrams:



The sum is free of Infrared Divercences.

The residual set of diagrams form the non-QED or weak corrections.

The ideal working strategy is to write the total amplitude as the sum of *dressed* γ and Z exchange amplitudes plus the contribution from *weak box* diagrams, i.e ZZ and WW boxes.

Fermionic loops are separately gauge invarint and may be re-summed. Bosonic loops have to be understood as expanded to first order.

One loop corrections and diagrams for $e^+e^- \to f\overline{f}$

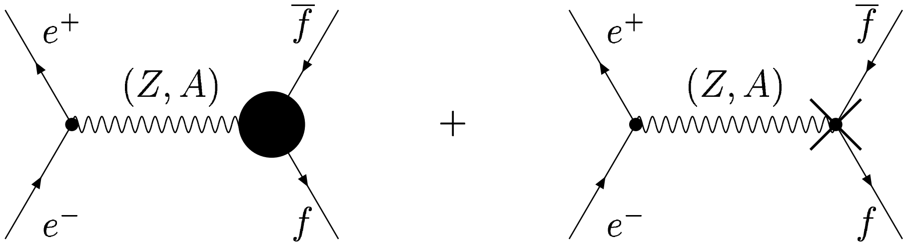


Figure 14: Process $e^+e^- \to (Z,A) \to f\overline{f}$; final fermion vertex and its counterterms

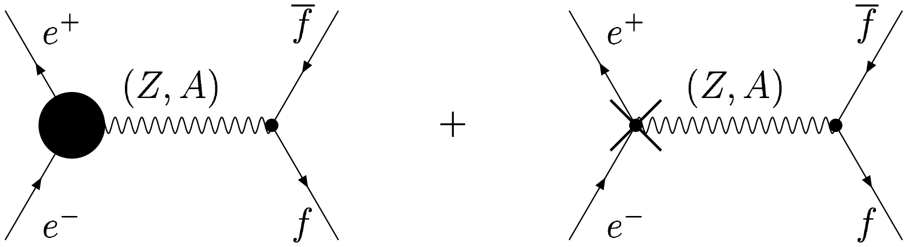


Figure 15: Process $e^+e^- \to (Z,A) \to f\overline{f}$; electron vertex and its counterterms

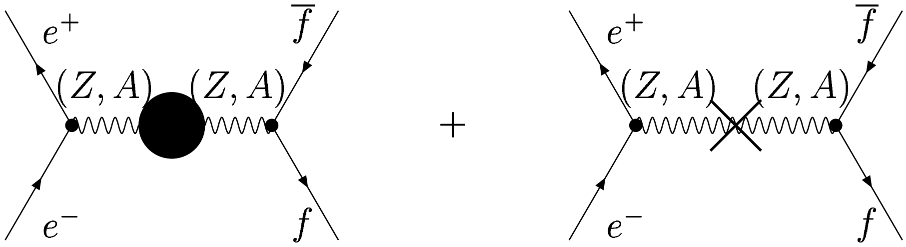


Figure 16: Process $e^+e^- \to (Z,A) \to f\overline{f}$; self energies and kinetic counterterms

If external fermion masses are neglected, the complete one loop amplitude (OLA) can be described by only four scalar functions and by running electromagnetic constant $\alpha^{\rm fer}(s)$.

Two ways for representing the dressed amplitude:

– in terms of four scalar form factors, $F_{ij}\,(s,t)$

$$\begin{aligned}
\mathcal{A}^{\rm OLA}_{Z+A} &= \frac{e^2 I^{(3)}_e I^{(3)}_f}{4 s^2_W c^2_W}\chi_Z(s)\Big\{\gamma_\mu\gamma_+ \otimes \gamma_\mu\gamma_+ F_{LL}\,(s,t) \\
&\quad -4|Q_e| s^2_W \gamma_\mu \otimes \gamma_\mu\gamma_+ F_{QL}\,(s,t) - 4|Q_f| s^2_W \gamma_\mu\gamma_+ \otimes \gamma_\mu F_{LQ}\,(s,t) \\
&\quad +16|Q_e Q_f| s^4_W \gamma_\mu \otimes \gamma_\mu F_{QQ}\,(s,t)\Big\}
\end{aligned}$$

– in terms of the effective couplings $\rho_{ef}(s,t)$ and $\kappa_{ij}(s,t)$

$$\begin{aligned}
\mathcal{A}^{\rm OLA}_{Z+A} &= \sqrt{2}G_F I^{(3)}_e I^{(3)}_f M^2_Z \chi_Z(s)\rho_{ef}(s,t)\Big\{\gamma_\mu\gamma_+ \otimes \gamma_\mu\gamma_+ \\
& -4|Q_e|s^2_W\kappa_e(s,t)\gamma_\mu \otimes \gamma_\mu\gamma_+ - 4|Q_f|s^2_W\kappa_f(s,t)\gamma_\mu\gamma_+ \otimes \gamma_\mu \\
& +16|Q_eQ_f|s^4_W\kappa_{ef}(s,t)\gamma_\mu \otimes \gamma_\mu\Big\}
\end{aligned}$$

On top of the $\mathcal{A}^{\rm OLA}_{Z+A}$ there is the corrected γ-exchange amplitude, which contains, by construction, only the QED running coupling $\alpha^{\rm fer}(s)$

$$\mathcal{A}^{\rm OLA}_{\gamma} = \frac{4\pi\alpha^{\rm fer}(s)}{s}\gamma_\mu \otimes \gamma_\mu$$

There are residual corrections to the photon exchange diagram but it is always possible to assign them to the Z exchange amplitude, since both contain the same Dirac structure $\gamma_\mu \otimes \gamma_\mu$.

Effective couplings ρ and κ's are related to the form factors and to the quantity Δr

$$\begin{aligned}
\rho_{ef} &= 1 + \frac{\alpha}{4\pi s^2_W}\Big[F_{LL}(s,t) - s^2_W\Delta r\Big] \\
\kappa_e &= 1 + \frac{\alpha}{4\pi s^2_W}\Big[F_{QL}(s,t) - F_{LL}(s,t)\Big] \\
\kappa_f &= 1 + \frac{\alpha}{4\pi s^2_W}\Big[F_{LQ}(s,t) - F_{LL}(s,t)\Big] \\
\kappa_{ef} &= 1 + \frac{\alpha}{4\pi s^2_W}\Big[F_{QQ}(s,t) - F_{LL}(s,t)\Big]
\end{aligned}$$

Here “1” is due to the Born amplitude which is added.

Higher order corrections or beyond one loop

EW and mixed corrections

EW two loop leading and next-to-leading corrections

$\Delta\rho,\ \Gamma\left(Z\to b\bar{b}\right),\quad \mathcal{O}\left(G_F^2 m_t^4\right),\quad$ *Barbieri et al. 92*

All observables but $\Gamma\left(Z\to b\bar{b}\right),\quad \mathcal{O}\left(G_F^2 m_t^2 M_Z^2\right),\quad$ *Degrassi et al. 96-98*

Implementation into ZFITTER, *DB and Degrassi, February 98*

Mixed two loop QCD-weak corrections

$\Delta\rho,\quad \mathcal{O}\left(\alpha\alpha_S\right),\quad$ *Djouadi at al. 88*

All observables, *DB and A.Chizhov 89 ; B.Kniehl 90*

$\Gamma\left(Z\to b\bar{b}\right),\quad \mathcal{O}\left(G_F\alpha_S m_t^2\right),\quad$ *FTJR, Fleisher et al. 93*

Leading three loop corrections to $\Delta\rho$

$$\Delta\rho,\quad \mathcal{O}\left(G_F\alpha_S^2 m_t^2\right),\quad \textit{AFMT, Avdeev et al. 94}$$

Purely QED corrections

Structure Functions (SF) and Flux Function (FF) languages.

$$\sigma\left(s\right)=\int_0^{1-s_0/s} dx H\left(x;s\right)\hat{\sigma}(\left(1-x\right)s)$$

where the function H, the so called *radiator* (or *flux function*), reads

$$\begin{aligned}
H\left(x;s\right) &= \int_{1-x}^{1}\frac{dz}{z}D\left(z;s\right)D\left(\frac{1-x}{z};s\right)\\
H\left(x;s\right) &= \beta\,x^{\beta-1}\,\delta^{\mathrm{V+S}}+\delta^{\mathrm{H}}\\
\beta &= \frac{2\,\alpha}{\pi}\left(L-1\right),\qquad L=\ln\frac{s}{m_e^2}
\end{aligned}$$

is known up to $\mathcal{O}\left(\alpha^2\right)$ *completely*, and $\mathcal{O}\left(\alpha^3 L^3\right)$ in *LLA*.

Deserve separate lecture.

QCD, mixed QED×QCD and mixed EW×QCD corrections

A lot of results are known, for vector and axial correction factors $R^f_{V,A}$, introduced above. Deserve separate course of lectures.

Precision calculations for LEP1 – swang-song of analytic approach!

Experimental Status of the Standard Model Outlook and Conclusions

Variety of LEP1 observables

Line shape → fig.1

$\sin^2\theta_{\rm eff}^{\rm lept}$ → fig.2

R_b versus R_c → fig.3

Pulls and χ^2/dof → fig.4

SM interpretation of precision data → fig.5

LEP1 teams still working on completion of analysis
SLAC, SLD asked for more running time, not yet approved

Variety of LEP2 and non-LEP observables

LEP2 amazing performance → fig.6

LEP2 example, $\sigma\left(e^+e^- \to W^+W^-\right)$ → fig.7

W mass summary → fig.8

M_W versus m_t → fig.9 and perspectives

LEP2 two years of measurements more, 1999-2000
500 pb^{-1} / experiment → $\Delta M_W = 30$ MeV
FNAL Run II starting year 2000, 2 pb^{-1} → $\Delta M_W = 40$ MeV
Serious hopes that ΔM_W will be mesured *directly*
more precise than it is presently known *indirectly*

BNL, E821, a_μ experiment → fig.10

M_H $\Delta\chi^2$ plot → fig.11 and perspectives

Still waiting for LEP2 last word
Linac 2×200 would do a good job

Conclusions

The Standard Model is completed
theoretically
and has to be ranked as The Standard Theory
which is supposed to completely replace QED

The Standard Theory is not completed
experimentally.
The Higgs boson is its only ingredient
which is still waiting for the discovery,
and it will be inevitably discovered

Where? When?

- Present members of BRG:
  DB, Lida Kalinovskaya, Penka Christova, Gizo Nanava, Anton Andonov
- Fields of interests:
  PHEP,
  Calculus for Modern and Future Colliders
- Resent and present work:
  - Finishing of works within ZFITTER project for LEP1-SLC-LEP2. (In close collaboration with Zeuthen group guided by Tord Riemann. A description of ZFITTER is published.)
  - Participation in just finished LEP2MCWS (1999–2000), last workshop on physics at LEP. (CERN Yelow Report is published.)
  - Creation of BRG-site with `mail goals`:
    * Database ordering of everything done in the field, by DZRCG and by DB and Giampiero Passarino while working on the book (`form` book support);
    * Creating the best environment to accomodate results that are being presently obtained and nearest future results;
    * Creating an environment for potential newcomers, preferrably diploma and post graduate students;
- Nearest work (still R&D phase):
  - Quest for concrete tasks (possibly LC oriented);
  - Quest for partners.